Resolve the multitude of microscale interactions to holistically discretise the stochastically forced Burgers' partial differential equation

A. J. Roberts*

May 22, 2006

Abstract

Constructing discrete models of stochastic partial differential equations is very delicate. Here we use modern dynamical systems theory to derive spatial discretisations of the nonlinear advection-diffusion dynamics of the stochastically forced Burgers' partial differential equation. In a region of the domain far from any spatial boundaries, stochastic centre manifold theory supports a discrete model for the dynamics. The trick to the application of the theory is to divide the physical domain into finite sized elements by introducing insulating internal boundaries which are later removed to fully couple the dynamical interactions between neighbouring elements. Burgers' equation is used as an example. The approach automatically parametrises the microscale, subgrid structures within each element induced by spatially varying stochastic forcing. The crucial aspect of this work is that we explore how a multitude of noise processes interact via the nonlinear dynamics within and between neighbouring elements. Noise processes with coarse structure across a finite element are the most significant noises for the discrete model. Their influence also diffuses away to weakly correlate the noise in the spatial discretisation. Further, the nonlinearity in the dynamics has two consequences: the example additive forcing generates multiplicative noise effects in the discretisation;

*Computational Engineering and Science Research Centre, Department of Mathematics & Computing, University of Southern Queensland, Toowoomba, Queensland 4352, Australia. mailto:aroberts@usq.edu.au

and effectively new noise sources are abstracted over the macroscopic time scales resolved by the discretisation. The techniques and theory developed here may be applied to discretise many dissipative stochastic partial differential equations.

Contents

1 Introduction

I introduce a dynamical systems approach to constructing discrete models of stochastic partial differential equations (SPDEs) by illustrating the concepts, analysis and results for the definite example of the stochastically forced Burgers' equation. The aim is to use dynamical systems theory and techniques to ensure the accuracy, stability and efficiency of numerical discretisations of SPDEs.

Furthermore, the sound methodology for modelling SPDEs presented here is likely to be needed to underpin multiscale modelling of physical systems in future applications [15, e.g.]. For example, the gap-tooth scheme of Kevrekidis et al. [22, 53, 54] is often implemented with particle simulators which are inherently stochastic within each simulation element. Hence connecting elements with stochastic microscale dynamics is as important as connecting elements with deterministic dynamics [50].

The non-dimensional stochastically forced Burgers' equation

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \sigma \phi(x, t) \,, \tag{1}$$

for a field $u(x, t)$ evolving in time t in one spatial dimension is a prototype example for many physically important SPDEs. The Burgers' SPDE (1) includes the mechanisms of dissipation, u_{xx}, nonlinear advection/steepening, uu_x, with nonlinearity parameter α, and the stochastic forcing $\phi(x, t)$ with strength parameter σ. Blömker et al. [4] analogously explored the rigorous modelling of the stochastically forced Swift–Hohenberg equation by a stochastic Ginzburg–Landau equation as a prototype SPDE in a class of pattern forming stochastic systems. Due to the forcing over many length and time scales, a SPDE typically has intricate spatio-temporal dynamics. Numerical methods to integrate stochastic *ordinary* differential equations are known to be delicate and subtle [26, e.g.]. We surely need to take considerable care for SPDEs as well [23, 59, e.g.].

For example, consider the forced diffusion equation obtained from the Burgers' SPDE (1) with nonlinearity parameter $\alpha = 0$. The simplest finite difference approximation in space on a regular grid in x, say $X_j = jh$ for some constant grid spacing h, is

$$\dot{U}_j = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + \sigma\phi(X_j, t)\,,$$

where the overdot denotes the derivative d/dt, and U_j is the value of the field $u(x,t)$ at the grid points X_j. However, the analysis of Section 3 recommends we use instead

$$\dot{U}_j \approx \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + \sigma\left[\sqrt{\tfrac{5}{7}}\psi_j - \tfrac{1}{24}\sqrt{\tfrac{7}{5}}(\psi_{j+1} - 2\psi_j + \psi_{j-1}) + \sqrt{\tfrac{2}{7}}\hat{\psi}_j\right], \tag{2}$$

see (31), for some noise processes ψ_j and $\hat{\psi}_j$. The rationale is that the spatial diffusion on the subgrid scale between the grid points weakly correlates the noise that should be applied to each grid value. Thus the point sample ϕ_j of the noise ϕ should be replaced by two components: a component $\hat{\psi}_j$ which is uncorrelated across the grid points; and a component ψ_j which has an influence distributed over the evolution of three neighbouring grid values. In order to find these interactions between noise and diffusion, we account explicitly for subgrid scale physical processes.

Centre manifold theory has wonderful characteristics for creating low dimensional models of dynamical systems (see the book by Carr [8] for a good introduction). It addresses the evolution of a dynamical system in a neighbourhood of a marginally stable fixed point; based upon the linear dynamics the theory guarantees that an accurate low-dimensional description of the nonlinear dynamics may be deduced. The theory is a powerful tool for the modelling of complex dynamical systems [9, 10, 32, 37, 21, e.g.] such as dispersion [52, 31, 58, e.g.], thin fluid films [11, 38, 41, e.g.] and other applications discussed in the review [39]. I apply the stochastic centre manifold theory of Boxler [5] to the discretisation of Burgers' SPDE (1).

1.1 Divide space into discrete finite elements

The method of lines discretises a PDE in space x and integrates in time as a set of ordinary differential equations, sometimes called a semi-discrete scheme [19, 20, e.g.]. Similarly, we only discuss the spatial discretisation of the Burgers' SPDE (1) and treat the resulting set of stochastic ordinary differential equations, such as (2), as a continuous time, stochastic dynamical system.

Place the discretisation of the nonlinear Burgers' SPDE (1) within the purview of stochastic centre manifold theory by the following artifice. Let equi-spaced grid points at x_j be a distance h apart. Then the jth element is notionally $|x - X_j| < h/2$. Form the elements by introducing the artificial internal boundary conditions (IBCs)

$$u_j(X_{j\pm1}, t) - u_j(X_j, t) = \gamma\left[u_{j\pm1}(X_{j\pm1}, t) - u_j(X_j, t)\right], \qquad (3)$$

where $u_j(x, t)$ denotes the subgrid scale field of the jth element. The coupling parameter γ controls the flow of information between adjacent elements: when $\gamma = 0$, adjacent elements are decoupled; when $\gamma = 1$, the field in the jth element must extrapolate to the neighbouring elements' field at their grid point. I proved [44] these IBCs ensured discrete models are consistent with linear deterministic PDEs to high order in small element size h; all examples also show these IBCs produce high order consistency for the nonlinear dynamics but no proof yet exists.

1.2 Model nonlinear stochastic dynamics

Via the analysis of Sections 4 and 5, a low accuracy discrete model of the nonlinear dynamics of the stochastically forced Burgers' SPDE (1) is

$$\begin{aligned} \dot U_j \approx{}& \frac{1}{h^2}(1 + \tfrac{1}{12}\alpha^2 h^2 U_j^2)(U_{j+1} - 2U_j + U_{j-1}) - \alpha\frac{1}{2h}U_j(U_{j+1} - U_{j-1}) \\ &+ \sigma\left[\phi_{j,0} - \alpha\frac{2h}{\pi^2}\phi_{j,1}U_j - \alpha^2\frac{8h^2}{3\pi^4}\phi_{j,2}U_j^2\right] + .01643\alpha^2 h^2\sigma^2 U_j\,, \qquad (4) \end{aligned}$$

when the subgrid scale noise within each element is truncated to the first three Fourier modes:

$$\phi(x, t) = \phi_{j,0}(t) + \phi_{j,1}(t)\sin[\pi(x - X_j)/h] + \phi_{j,2}(t)\cos[2\pi(x - X_j)/h]\,.$$

The first line of the discretisation (4) is the so-called holistic discretisation for the deterministic Burgers' equation which has good properties on finite sized elements [42]; in particular, see that the nonlinearly enhanced diffusion enhances the stability of the scheme for non-small field u. The second line of the discretisation (4) approximates some of the influences of the forcing noise: observe that the nonlinearity in the subgrid scale dynamics of Burgers' equation transforms the additive noise forcing of the SPDE (1) into multiplicative noise components in the discretisation. Simple modelling schemes miss such multiplicative noise terms because they do not resolve the subgrid scale processes.

Stochastic forcing generates high wavenumber, steep variations, in spatial structures. Stable implicit time integration very rapidly damps such decaying modes, yet through a form of stochastic resonance a reasonably accurate resolution of the life-time of these modes may be important on the large scale dynamics. For example, the discrete model (4) includes a term proportional to $\sigma^2 U_j$ that arises from the self-interactions of noise, as flagged by the σ^2 factor and discussed in Section 5.3; here this term demonstrates that subgrid microscale noise interactions destabilise the equilibrium $u = 0$. Herein the term “stochastic resonance” includes phenomena where stochastic fluctuations interact with each other and themselves through nonlinearity in the dynamical system to generate not only long time drifts but also potentially to change stability [27, 5, 17, 47, 57, e.g.]. Consequently, numerical discretisation, such as (4), with large space-time grids must resolve subgrid microscale structures to achieve efficiency without sacrificing the subtle interactions that take place between the subgrid scale structures, such as those seen in the noise correlations in (2).

Centre manifold theory supports the large time macroscopic modelling of detailed stochastic microscopic dynamics. For example, Knobloch & Wiesenfeld [27] and Boxler [5, 6] explicitly used centre manifold theory to support the modelling of SDEs and SPDEs. Boxler [5] proves that “stochastic center manifolds, share all the nice properties of their deterministic counterparts.” Many, such as Berglund & Gentz [3], Blömker, Hairer & Pavliotis [4] and Kabanov & Pergamenshchikov [25], use the same separation of time scales that underlies the application of centre manifolds to form and support low-dimensional, long time models of SDEs and SPDEs that have both fast and slow modes. Centre manifold theory also supports the discretisation on finite sized grids of deterministic partial differential equations [42, 44, 28, 43, 45, 29]. By merging these two applications of centre manifold theory we model SPDEs with sound theoretical support, as described in Section 2. Coupling many finite elements together forms macroscale discrete models of SPDEs such as (2) and (4). The fiendish complication is to account for noise and its dynamics which are distributed independently across space as well as time, both within a finite element and between neighbouring finite elements. Computer algebra [49] handles the details of the nonlinear subgrid dynamics and the inter-element interactions.

Note three aspects throughout this work.

- We discuss the forcing $\phi(x,t)$ as a white noise, that is, $\phi(x,t)$ is delta correlated in both space and time. Although computational limitation often require the truncation to a few Fourier modes as in (4). However, most of the analysis and models in Sections 2–4 also hold for deterministic forcing $\phi(x,t)$.
- Interpret all noise processes and all stochastic differential equations in the Stratonovich sense so that the rules of traditional calculus apply. Thus the direct application of this modelling is to physical systems where the Stratonovich interpretation is the norm.
- Consider the Burgers' SPDE (1) in the interior of a domain large enough so the boundaries are far enough away to be immaterial. Equivalently, the analysis may be in a domain with periodic boundary conditions. The crucial aspect is that the analysis throughout assumes space and time are homogeneous (the stochastic forcing $\phi(x,t)$ is statistically homogeneous) so that the resultant discretisations are homogeneous as seen in (2) and (4).

2 Centre manifold theory underpins modelling

I detail one way to place the spatial discretisation of SPDEs within the purview of stochastic centre manifold theory. Then the theory assures us of the existence and relevance of the discrete models constructed in later sections. The stochastic forced Burgers' equation (1) serves as a definite example of a broad class of nonlinear, dissipative SPDEs.

We *base* the discrete modelling upon the dynamics when: firstly, the noise is absent, $\sigma = 0$; secondly, each element is decoupled from its neighbours, $\gamma = 0$; and lastly, the nonlinearity is negligible, $\alpha = 0$. When $\sigma = \gamma = \alpha = 0$ the dynamics of Burgers' SPDE (1) with coupling conditions (3) reduce to that of linear diffusion within each element insulated from its neighbours.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{such that} \quad u(X_{j\pm1},t) = u(X_j,t)\,. \tag{5}$$

The discrete coupling conditions (3) and their linearisation in (5) are most convenient for the construction of the discrete model. However, theory is most conveniently applied using the equivalent coupling conditions

$$\pm h\frac{\partial u_j}{\partial x}\bigg|_{x=X_j\pm h/2} = \gamma\left[\mathcal{A}u_{j\pm1}|_{x=X_{j\pm1}} - \mathcal{A}u_j|_{x=X_j}\right], \tag{6}$$

where the near identity operator

$$\mathcal{A} = \frac{\pm h\partial_x}{\exp(\pm h\partial_x)-1} = 1 \mp \tfrac{1}{2}\partial_x + \tfrac{1}{12}\partial_x^2 - \tfrac{1}{720}\partial_x^4 + \tfrac{1}{30240}\partial_x^6 + \mathcal{O}\big(\partial_x^8\big).$$

All discussions of theoretical support for the discretisation use the element coupling conditions (6) instead of the computationally convenient (3).

The theory also needs a definite domain. Thus all discussions of theoretical support use a domain in space coordinate x of some length L. The boundary conditions are that the field $u(x,t)$ is to be L periodic in x. Divide the domain into M elements of equal and finite length $h = L/M$. The grid point $x = X_j$ is the mid-point of the jth element.

Using the coupling conditions (6), instead of (5) the base problem is the diffusion dynamics on each of the M elements with insulating boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{such that} \quad \pm h\frac{\partial u}{\partial x} = 0 \text{ at } x = X_j \pm \tfrac{1}{2}h. \tag{7}$$

This base problem has solution in each element composed of modes $u \propto \operatorname{csn} k\theta \exp(-\beta_k t)$ where: firstly, the Fourier modes

$$\operatorname{csn} k\theta = \begin{cases} \cos k\theta, & \text{for even } k, \\ \sin k\theta, & \text{for odd } k; \end{cases} \tag{8}$$

secondly, the integer k is the subgrid scale wavenumber; thirdly, $\theta = \pi(x - X_j)/h$ measures subgrid position relative to the grid point within each element—the jth element lies between $\theta = \pm\pi/2$; and lastly, the kth mode decays with rate

$$\beta_k = \frac{\pi^2 k^2}{h^2}, \quad k = 0, 1, 2, \ldots. \tag{9}$$

The $k = 0$ mode, u is constant in each element, is linearly neutral as its decay rate $\beta_0 = 0$, and thus forms the basis of the long term model.

Decompose the noise within each element as a linear combination of the fundamental Fourier modes (8):

$$\phi(x,t) = \sum_{k=0}^{\infty} \phi_{j,k}(t) \operatorname{csn} k\theta, \quad \text{for } |x - X_j| < h/2, \tag{10}$$

where $\phi_{j,k}$ denotes the noise process of the kth wavenumber in the jth element. Assume the set of processes $\{\phi_{j,k}\}$ are independent. The components of the forcing noise (10) with wavenumber $k \geq 1$ are orthogonal to the neutral basic mode of the field u being constant in each element. Consequently, simple numerical methods, such as Galerkin projection onto the basic mode, would ignore the "high wavenumber" modes, $k \geq 1$, of the noise (10) and hence completely miss subtle but important subgrid and inter-element interactions such as those seen in the models (2) and (4). Instead, the systematic nature of centre manifold theory accounts for the subgrid scale interactions as a power series in the noise amplitude σ, the inter-element coupling γ and the nonlinearity α from the deterministic base (7).

A stochastic centre manifold exists The nonlinear forced Burgers' SPDE (1) with inter-element coupling conditions (6) linearises to the diffusionPDE (7). This linear PDE has M eigenvalues of zero and all the other eigenvalues are negative, namely

$$\text{the stable eigenvalues} \leq -\beta_1 = -\pi^2/h^2 . \tag{11}$$

After adjoining the three trivial DEs $d\boldsymbol{\epsilon}/dt = 0$, where $\boldsymbol{\epsilon} = (\sigma, \gamma, \alpha)$, stochastic centre manifold theory [5, Theorem 5.1 and 6.1] assures us that in some finite neighbourhood of $(u, \boldsymbol{\epsilon}) = (0, \mathbf{0})$ there exists an $M + 3$ dimensional stochastic centre manifold where the field in the jth element is $u = u_j(\mathbf{U}(t), x, t, \boldsymbol{\epsilon})$ where the jth component U_j of vector $\mathbf{U}$ measures the amplitude of the neutral mode in the jth element. For example, we endeavour to define the amplitude

$$U_j = u_j(\mathbf{U}, X_j, t, \boldsymbol{\epsilon}) , \tag{12}$$

then an example low accuracy approximation to the centre manifold is simply the deterministic Lagrangian interpolation within each element:

$$u_j = U_j + \gamma \tfrac{1}{2}(\theta/\pi)(U_{j+1} - U_{j-1}) + \gamma \tfrac{1}{2}(\theta/\pi)^2(U_{j+1} - 2U_j + U_{j-1}) + \cdots . \tag{13}$$

Of the dimensions of the stochastic centre manifold, M dimensions arise from the one neutral mode within each of the M elements, and three dimensions arise from the dependence upon the three parameters $\boldsymbol{\epsilon} = (\sigma, \gamma, \alpha)$. On the stochastic centre manifold the amplitudes U_j evolve according to $\dot{U}_j = g_j(\mathbf{U}, t, \boldsymbol{\epsilon})$ for some function g_j. Unfortunately, there is a caveat: Boxler's [5] theory is so far developed only for finite dimensional systems which satisfy a Lipschitz condition. Here, the SPDE (1) is infinite dimensional and the nonlinear advection uu_x involves the unbounded operator $\partial/\partial x$. Nonetheless, Blömker et al. [4, Theorem 1.2] rigorously proved the existence and relevance of a stochastic Ginzburg–Landau model to the 'infinite dimensional' stochastic forced Swift–Hohenberg PDE; further, Caraballo, Langa & Robinson [7] and Duan, Lu & Schmalfuss [18] proved the existence of invariant manifolds for a wide class of 'infinite dimensional' reaction-diffusion SPDEs; they built on earlier work on inertial manifolds in SPDEs by Bensoussan & Flandoli [2]. Future theoretical developments should rigorously support this approach.

However, in the interim, a way to proceed is via a type of shadowing argument [48]. The rapid dissipation of high wavenumber modes in (1), the spectrum $\lambda = -\pi^2 k^2/h^2$ for wavenumber k, ensures that the dynamics of the SPDE (1) is close to finite dimensional. By modifying the spatial derivatives in (1) to have a high wavenumber cutoff, the dynamics of the SPDE (1) is effectively that of a Lipschitz, finite dimensional system. The theorems of Boxler [5] then rigorously apply. Indeed, in constructing the centre manifold model of the nonlinear dynamics of Burgers' SPDE (1), in Section 4, I am compelled by computational limitations to resort to projecting the dynamics within each element onto a finite number of Fourier modes. This projection immediately transforms Burgers' SPDE (1) into a finite dimensional, Lipschitz system to which Boxler's [5] theory applies. That is, when we analyse the difficult nonlinear dynamics, the projection onto a nearby system by the Fourier truncation also enables rigorous theoretical support.

The centre manifold model captures the dynamics The second key theorem of stochastic centre manifolds is that the evolution on the centre manifold, such as that described by (2) or (4), do capture the long term dynamics of the original stochastic SPDE (1). The Stichastic Relevance Theorem 7.1(i) [5] assures us that all nearby solutions of the SPDE (1) exponentially quickly in time approach the stochastic centre manifold $u = u_j(\mathbf{U}(t), x, t, \epsilon)$, such as the low accuracy approximation (13). But crucially the theorem also guarantees that the evolution of the trajectories approaching the centre manifold also approaches exponentially quickly the evolution of a trajectory on the centre manifold. That is, the evolution on the stochastic centre manifold, such as (2) or (4), faithfully describes the evolution of *all solutions* of Burgers' SPDE (1) in some neighbourhood of the stochastic centre manifold. (This property has been called "asymptotic completeness" [51] in deterministic systems.) Thus, in this context, the centre manifold model forms a discrete model that describes all the dynamics of Burgers' SPDE (1) apart from exponentially decaying transients. This amazing theoretical support for the model holds at finite element size h.

The Stochastic Relevance Theorem 7.1(i) [5] also asserts that the rate of decay to the centre manifold may be estimated by the gravest subgrid scale mode, here $\sin\theta\,\exp(-\beta_1 t)$. Thus on times significantly larger than a cross element diffusion time h^2/π^2, the exponential transients decay and the centre manifold model describes the dynamics of Burgers' SPDE (1).

However, there are two significant caveats. Firstly, although the asymptotic series we do construct are global in the grid value amplitudes U_j, they are local in the parameters $\boldsymbol\epsilon = (\sigma, \gamma, \alpha)$. Thus the rigorous theoretical support only applies in some finite neighbourhood of $\boldsymbol\epsilon = \mathbf{0}$. At this stage we have little information on the size of that neighbourhood. In particular we need to evaluate the model when $\gamma = 1$ to recover a model for when the elements are fully coupled together; thus we desire $\gamma = 1$ to be in the finite neighbourhood of validity. This has been demonstrated for the deterministic Burgers' equation [42], but not yet for the stochastic case. Secondly, we cannot construct the stochastic centre manifold and the evolution thereon exactly; it is difficult enough constructing asymptotic approximations such as the low order accuracy models (2) and (4). Thus the models we develop and discuss have an error due to the finite truncation of the asymptotic series in the small parameters $\boldsymbol\epsilon$.

For example, the truncation in powers of the coupling parameter γ controls the width of the computational stencil for the discrete models. The communication between adjacent elements occurs through the coupling conditions (3) or (6) moderated by the coupling parameter γ. Thus nearest neighbour elements interactions are flagged by terms in γ^1, whereas interactions with next to nearest neighbouring elements are occur as γ^2 terms, and so on for higher powers. The low accuracy models (2) and (4) are constructed with error $\mathcal{O}(\gamma^2)$ and so summarise the interactions between the dynamics in an element and those of its two immediate neighbours.

In a nonlinear system the noise processes interact with each other and themselves. Such interactions generate mean drift effects explored in Section 5. Thus Section 4 computes asymptotic solutions of Burgers' SPDE (1) to errors $\mathcal{O}(\sigma^3)$ so we retain the crucial self-interaction noise terms in σ^2. But dealing with nonlinear stochastic dynamics is very complicated and so Section 3 first introduces some of the techniques in the considerably simpler example of stochastically forced diffusion.

3 Construct a discrete model of diffusion in a memoryless normal form

This section explores the discretisation of the non-dimensional stochastically forced diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sigma\phi(x,t)\,. \tag{14}$$

This forced diffusion dynamics is the special case of Burgers' SPDE (1) when the nonlinearity parameter $\alpha = 0$.

The main reason for first investigating the dynamics of the SPDE (14) is that the dynamics are linear. Many of the methods are significantly easier to explore in this linear diffusion instead of the nonlinear Burgers' dynamics. We explore how to construct a stochastic centre manifold model of the discretised dynamics. The resulting models show how diffusion of noise processes within and between adjacent elements spatially correlate noise in the discretisation as seen in (2).

3.1 Iteration converges to the asymptotic series

The centre manifold approach establishes that the long term dynamics of a SPDE such as Burgers' equation (1) or the diffusion equation (14) may be parametrised by a measure of the solution in each element, here called U_j. Arnold et al. [1] investigated stochastic Hopf bifurcations this way, and Schoner & Haken's [55] slaving principle is equivalent. However, most modellers of SDEs generate models with convolutions over fast time scales of the noise [47, §2, e.g.]. Here we simplify the model tremendously by removing such 'memory' convolutions as originally developed for SDEs by Coullet, Elphick & Tirapegui [13], Sri Namachchivaya & Lin [35], and Roberts & Chao [12, 47].

As discussed in Section 2, centre manifold theory supports the discretisation. The construction theorem only depends upon finding a model for which the residuals of the governing equations are of some specified order of smallness: Boxler [5, Theorem 8.1] assures us that if we satisfy the SPDE (14) to some residual $\mathcal{O}(\epsilon^q)$, then the stochastic centre manifold and the evolution thereon have the same order of error, namely $\mathcal{O}(\epsilon^q)$. Recall that the noise intensity σ and the inter-element coupling γ are the small parameters in the asymptotic series forming the model of the diffusion. Because the critical aspect of constructing the centre manifold model is simply the ultimate order of the residual of the SPDE (14), the specific details of the computation are not recorded here. Instead computer algebra [49] performs all the details. Here I report on critical steps in the method.

Consider the task of iteratively constructing [40] a stochastic model for the SPDE (14). We seek solutions such that in the jth element the field $u = u_j(\mathbf{U}, x, t, \boldsymbol{\epsilon}) = U_j + \cdots$ such that the vector of amplitudes $\mathbf{U}$ evolve according to some prescription $\dot{U}_j = g_j(\mathbf{U}, t, \boldsymbol{\epsilon})$, such as (2). The steps in the construction proceed iteratively. Suppose that at some stage we have some asymptotic approximation to the model, then the next iteration is to seek small corrections, denoted u'_j and g'_j, to improve the asymptotic approximation. As the iterations proceed, the small corrections u'_j and g'_j get systematically smaller, that is, of higher order in the small parameters $\boldsymbol{\epsilon}$ of the asymptotic series. As explained in [40], substitute $u = u_j + u'_j$ and $\dot{U}_j = g_j + g'_j$ into the SPDE (14), then linearise the problem for u'_j and g'_j by dropping products of small corrections. Thus obtain that the corrections should satisfy

$$\frac{\partial u'_j}{\partial t} - \frac{\partial^2 u'_j}{\partial x^2} + g'_j = \text{residual}_{(14)}. \tag{15}$$

Here the "residual" is the residual of the SPDE (14) evaluated for the currently known asymptotic approximation. In addition, the inter-element coupling conditions (3) provide boundary conditions for u'_j: substitute $u = u_j + u'_j$ into (3); linearise by dropping small $\gamma u'_j$ terms; and obtain that the correction PDE (15) needs to be solved with the boundary conditions

$$u'_j(X_{j\pm1}, t) - u'_j(X_j, t) + \text{residual}_{(3)} = 0\,. \tag{16}$$

For example, suppose at some stage we had found the deterministic part of the model in the jth element was that of classic Lagrangian interpolation

$$\begin{aligned} & u_j(x,t) = U_j + \gamma\left[\tfrac{1}{2}(\theta/\pi)^2\delta^2 + (\theta/\pi)\mu\delta\right]U_j + \mathcal{O}\big(\sigma+\gamma^2\big) \\ \text{such that}\quad & \dot{U}_j = \frac{\gamma}{h^2}\delta^2 U_j + \mathcal{O}\big(\sigma+\gamma^2\big)\,, \end{aligned} \tag{17}$$

where throughout this article the discrete difference and mean operators [36, p.65, e.g.] reduce the algebraic length of expressions, respectively

$$\delta U_j = U_{j+1/2} - U_{j-1/2} \quad\text{and}\quad \mu U_j = \tfrac{1}{2}\left(U_{j+1/2} + U_{j-1/2}\right)\,.$$

Then using (17) in the next iteration, the diffusion equation's

$$\text{residual}_{(14)} = -\frac{\gamma^2}{h^2}\left[(\theta/\pi)\mu\delta^3 + \tfrac{1}{2}(\theta/\pi)^2\delta^4\right]U_j + \sigma\sum_{k=0}^{\infty}\phi_{j,k}(t)\operatorname{csn}k\theta\,, \tag{18}$$

whereas the coupling conditions have

$$\text{residual}_{(3)} = 0\,. \tag{19}$$

3.2 Corrections from a simple residual

Now explore how to solve (15–16) for corrections given some residual such as (18–3). The terms in the residual split into two categories, as is standard in singular perturbations:

- Each component in $\operatorname{csn} k\theta$ for $k \geq 1$ causes no great difficulty; we include a corresponding component in the correction u_j' to the field in proportion to $\operatorname{csn} k\theta$—when the coefficient of $\operatorname{csn} k\theta$ in the residual is time dependent the component in the correction u_j' is $\mathcal{Z}_k\phi_{j,k}(t)\operatorname{csn} k\theta$ in which the operator $\mathcal{Z}_k$ denotes convolution over past history with $\exp[-\beta_k t]$, namely

$$\mathcal{Z}_k\phi = \exp[-\beta_k t] \star \phi(t) = \int_{-\infty}^{t} \exp[-\beta_k(t-\tau)]\phi(\tau)\,d\tau\,; \tag{20}$$

  recall that $\beta_k = k^2\pi^2/h^2$ is the (positive) decay rate (9) of the kth mode within each element.

- But any component constant across the element, such as $\sigma\phi_{j,0}(t)$ in this iteration with residual (18), must cause a contribution to the evolution correction g_j', here simply $g_j' = \sigma\phi_{j,0}$, as no uniformly bounded component in u_j' can match a constant component of the residual—this is the standard solvability condition for singular perturbations.

- Note that, as in previous research on deterministic systems [40, 44, e.g.], the deterministic part of the residual is decomposed into the above two components without explicitly invoking the Fourier transform.

For example, with the residuals (18–19) the corresponding corrections g_j' and u_j' improve (17) to

$$\begin{aligned} u_j(x,t) &= U_j + \gamma\left[\tfrac{1}{2}(\theta/\pi)^2\delta^2 + (\theta/\pi)\mu\delta\right]U_j \\ &\quad + \gamma^2\left[\tfrac{1}{6}((\theta/\pi)^3 - (\theta/\pi))\mu\delta^3 + \tfrac{1}{24}((\theta/\pi)^4 - (\theta/\pi)^2)\delta^4\right]U_j \\ &\quad + \sigma\sum_{k=1}^{\infty}\mathcal{Z}_k\phi_{j,k}\operatorname{csn} k\theta \quad + \mathcal{O}\big(\sigma^{3/2} + \gamma^3\big)\,, \end{aligned} \tag{21}$$

$$\dot{U}_j = \frac{\gamma}{h^2}\delta^2 U_j - \frac{\gamma^2}{12h^2}\delta^4 U_j + \sigma\phi_{j,0} + \mathcal{O}\big(\sigma^{3/2} + \gamma^3\big)\,, \qquad (22)$$

The γ^2 corrections modify the deterministic terms of the model (17) to (when $\gamma = 1$) classic finite difference expressions of fourth order consistency as element size $h \to 0$; the noise induced σ terms are the straightforward forcing of the model dynamics. It is the next iteration that begins to account for interesting subgrid scale stochastic processes within the finite sized elements.

Let the amplitude be flexible Observe that due to the above correction u_j', the field evaluated at the grid points $x = X_j$ is no longer the amplitude U_j as we initially requested by (12); instead, with this correction the grid value $u(X_j, t) = U_j + \sigma \sum_{k=2,\text{even}}^{\infty} \mathcal{Z}_k\phi_{j,k}$. Insisting that the evolution $\dot{U}_j = g_j'$ does not have any memory convolutions implies we cannot require the amplitudes U_j to be the grid values $u(X_j, t)$. We must abandon absolute control over the meaning of the amplitudes when modelling non-autonomous dynamical systems. In the geometry of state space, removing memory convolutions from the evolution *implicitly* requires that the amplitudes be constant along the so-called isochrons in the neighbourhood of the centre manifold [14].

3.3 Some convolutions need to be separated

A more delicate issue arises in subsequent corrections. The next iteration uses (21), whence the coupling conditions have

$$\text{residual}_{(3)} = \gamma\sigma \sum_{k=2,\text{even}}^{\infty} [\mathcal{Z}_k\phi_{j,k} - \mathcal{Z}_k\phi_{j\pm1,k}] + \mathcal{O}\big(\sigma^3 + \gamma^3\big)\,.$$

Satisfy the coupling conditions (16) with the above $\text{residual}_{(3)}$ by incorporating into the approximate field (21), the following correction u_j' (quadratic across the element)

$$+\gamma\sigma\left[(\theta/\pi)\mu\delta + \tfrac{1}{2}(\theta/\pi)^2\delta^2\right] \sum_{k=2,\text{even}}^{\infty} \mathcal{Z}_k\phi_{j,k}\,.$$

Then using (22) and the above added to the right-hand side of (21), the diffusion SPDE has

$$\text{residual}_{(14)} = \gamma\sigma\frac{\delta^2}{h^2} \sum_{k=2,\text{even}}^{\infty} \mathcal{Z}_k\phi_{j,k}$$

$$- \gamma\sigma \left[(\theta/\pi)\mu\delta + \tfrac{1}{2}(\theta/\pi)^2\delta^2\right] \left[\sum_{k=0,\text{even}}^{\infty} \phi_{j,k} - \sum_{k=2,\text{even}}^{\infty} \beta_k \mathcal{Z}_k \phi_{j,k}\right] + \mathcal{O}\big(\sigma^3 + \gamma^3\big) , \tag{23}$$

with coupling condition residual$_{(3)} = \mathcal{O}\big(\sigma^3+\gamma^3\big)$. The terms in (23) involve the interaction of noise terms with the inter-element coupling, flagged by $\sigma\gamma$. We proceed to find the correction they force through solving (15) which identifies how such a forcing affects the subgrid diffusion and time evolution. Because all of the terms in the residual (23) have fast time variations, through each of the noises $\phi_{j,k}$, decompose the subgrid spatial structure into the Fourier modes using

$$(\theta/\pi) = \frac{4}{\pi^2} \sum_{k=1,\text{odd}}^{\infty} \frac{(-1)^{\frac{k-1}{2}}}{k^2} \sin k\theta ,$$

$$\text{and} \qquad (\theta/\pi)^2 = \frac{1}{12} + \frac{4}{\pi^2} \sum_{k=2,\text{even}}^{\infty} \frac{(-1)^{k/2}}{k^2} \cos k\theta .$$

Then the above

$$\begin{aligned}
\text{residual}_{(14)} = {} & \gamma\sigma\delta^2 \sum_{k=2,\text{even}}^{\infty} \left(\frac{1}{h^2} + \frac{\beta_k}{24}\right) \mathcal{Z}_k \phi_{j,k} - \gamma\sigma\frac{1}{24}\delta^2 \sum_{k=0,\text{even}}^{\infty} \phi_{j,k} \\
& - \gamma\sigma\frac{4}{\pi^2} \sum_{k=1,\text{odd}}^{\infty} \frac{(-1)^{\frac{k-1}{2}}}{k^2} \sin k\theta\, \mu\delta \left[\sum_{\ell=0,\text{even}}^{\infty} \phi_{j,\ell} - \sum_{\ell=2,\text{even}}^{\infty} \beta_\ell \mathcal{Z}_\ell \phi_{j,\ell}\right] \\
& - \gamma\sigma\frac{2}{\pi^2} \sum_{k=2,\text{even}}^{\infty} \frac{(-1)^{k/2}}{k^2} \cos k\theta\, \delta^2 \left[\sum_{\ell=0,\text{even}}^{\infty} \phi_{j,\ell} - \sum_{\ell=2,\text{even}}^{\infty} \beta_\ell \mathcal{Z}_\ell \phi_{j,\ell}\right] \\
& + \mathcal{O}\big(\sigma^3 + \gamma^3\big) .
\end{aligned} \tag{24}$$

The components in csn $k\theta$ above are not an issue; they just induce a corresponding component in the correction u_j' via a further convolution $\mathcal{Z}_k$. The components constant across the element, in the first line above, are the delicate issue:

- we cannot match them by corrections u_j' as then u_j' would contain integrals in time of the noise processes which in general grow secularly like $\sqrt{t}$;
- neither can they be matched by corrections to the evolution g_j' as then incongruous fast-time convolution integrals would appear in the model of the long term dynamics.

The appropriate alternative [13, 35, 12, 47] recognises that part of these components can be integrated in time: since for any $\phi(t)$, $\frac{d}{dt}\mathcal{Z}_k\phi = -\beta_k\mathcal{Z}_k\phi + \phi$, from the convolution definition (20), thus

$$\mathcal{Z}_k\phi = \frac{1}{\beta_k}\left[-\frac{d}{dt}\mathcal{Z}_k\phi + \phi\right], \tag{25}$$

and so separate such a convolution in the residual, when multiplied by the neutral mode of a constant across the element, into:

- the first part, $-\frac{d}{dt}\mathcal{Z}_k\phi/\beta_k$, which is integrated into the next update u_j' for the subgrid field;
- and the second part, ϕ/β_k, which updates g_j' without introducing a fast-time memory convolution into the evolution.

For the example residual (24) the terms in the first line thus force terms

$$-\gamma\sigma\delta^2\sum_{k=2,\text{even}}^{\infty}\left(\frac{1}{\pi^2k^2}+\frac{1}{24}\right)\mathcal{Z}_k\phi_{j,k}$$

into the subgrid field making it now

$$\begin{aligned}
u_j(x,t) = {}& U_j + \gamma\left[\tfrac{1}{2}(\theta/\pi)^2\delta^2 + (\theta/\pi)\mu\delta\right]U_j \\
&+ \gamma^2\left[\tfrac{1}{6}((\theta/\pi)^3-(\theta/\pi))\mu\delta^3 + \tfrac{1}{24}((\theta/\pi)^4-(\theta/\pi)^2)\delta^4\right] \\
&+ \sigma\sum_{k=1}^{\infty}\mathcal{Z}_k\phi_{j,k}\operatorname{csn}k\theta \\
&- \gamma\sigma\frac{4}{\pi^2}\sum_{k=1,\text{odd}}^{\infty}\frac{(-1)^{\frac{k-1}{2}}}{k^2}\sin k\theta\,\mu\delta\left[\sum_{\ell=0,\text{even}}^{\infty}\mathcal{Z}_k\phi_{j,\ell} - \sum_{\ell=2,\text{even}}^{\infty}\beta_\ell\mathcal{Z}_{k,\ell}\phi_{j,\ell}\right] \\
&- \gamma\sigma\frac{2}{\pi^2}\sum_{k=2,\text{even}}^{\infty}\frac{(-1)^{k/2}}{k^2}\cos k\theta\,\delta^2\left[\sum_{\ell=0,\text{even}}^{\infty}\mathcal{Z}_k\phi_{j,\ell} - \sum_{\ell=2,\text{even}}^{\infty}\beta_\ell\mathcal{Z}_{k,\ell}\phi_{j,\ell}\right] \\
&- \gamma\sigma\delta^2\sum_{k=2,\text{even}}^{\infty}\left(\frac{1}{\pi^2k^2}+\frac{1}{24}\right)\mathcal{Z}_k\phi_{j,k} \quad + \mathcal{O}\big(\sigma^3+\gamma^3\big),
\end{aligned} \tag{26}$$

where $\mathcal{Z}_{k,\ell}$ denotes the two compounded convolutions $\mathcal{Z}_k\mathcal{Z}_\ell$. More interestingly, the terms in the first line of the example residual (24) also force the correction to the evolution

$$g_j' = \gamma\sigma\delta^2\left[-\frac{1}{24}\phi_{j,0} + \frac{1}{\pi^2}\sum_{k=2,\text{even}}^{\infty}\frac{1}{k^2}\phi_{j,k}\right]. \tag{27}$$

This separation, of the forcing constant across each element, ensures that the subgrid field u_j is always bounded, and the model evolution for the amplitudes U_j does not have any incongruous fast time convolutions.

When the residual component has many convolutions, then apply this separation recursively.

Continuing this iterative construction gives more and more accurate models. The iteration terminates when the residuals are zero to some specified order. Then the Approximation Theorem of centre manifold theory [5, Theorem 8.1] assures us that the model has the same order of error as the residual. For example, adding (27) to (22), the model evolution is

$$\begin{aligned} \dot{U}_j &= \frac{\gamma}{h^2}\delta^2 U_j - \frac{\gamma^2}{12h^2}\delta^4 U_j + \sigma\phi_{j,0} \\ &\quad + \gamma\sigma\delta^2\left[-\frac{1}{24}\phi_{j,0} + \frac{1}{\pi^2}\sum_{k=2,\text{even}}^{\infty}\frac{1}{k^2}\phi_{j,k}\right] + \mathcal{O}\big(\sigma^3+\gamma^3\big)\,. \end{aligned} \tag{28}$$

The order of error comes from the terms present in the residuals but so far ignored when determining corrections.

3.4 Diffusion correlates noise across space

Because simple diffusion is linear, there are no stochastic interaction terms, those of $\mathcal{O}(\sigma^2)$, in the model (28). But note that the noise applied to the jth grid value U_j is coupled to the noise sources in neighbouring elements through the second difference $\delta^2\phi_{j,k}$ terms. This coupling arises because the noise in one element creates spatial structures that diffuse out into neighbouring elements and affect the evolution. The coupling does not depend upon element size because the diffusion time into a neighbouring element has the same time scale as diffusive decay within each element.

The analysis so far applies whether the forcing $\phi(x,t)$ is deterministic or stochastic: (28) also models deterministic forcing. However, when the forcing components $\phi_{j,k}$ are independent stochastic processes then the model may be simplified as described in this and the next subsection.

The asymptotic approximation (28) models forced diffusion dynamics, (14), when we set the coupling parameter $\gamma = 1$. Undesirably, the resultant model has infinite sums of noise components:

$$\dot{u}_j = \frac{1}{h^2}\delta^2 u_j - \frac{1}{12h^2}\delta^4 u_j + \sigma\phi_{j,0} + \sigma\delta^2\left[-\frac{1}{24}\phi_{j,0} + \frac{1}{\pi^2}\sum_{k=2,\text{even}}^{\infty}\frac{1}{k^2}\phi_{j,k}\right]. \quad (29)$$

But these noises are unknown. Thus we may combine the infinite sums of noise terms into new unknown noises with the same statistics as the infinite sums. Let us explore two more different versions.

1. The combination

$$\frac{1}{\pi^2}\sum_{k=2,\text{even}}^{\infty}\frac{1}{k^2}\phi_{j,k} \equiv \frac{1}{4\pi^2}\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^4}}\,\hat{\phi}_j(t) = \frac{1}{12\sqrt{10}}\hat{\phi}_j(t)$$

   where the effectively new stochastic noise $\hat{\phi}_j(t)$ represents the cumulative effect of the infinite sum of the stochastic components $\phi_{j,k}$ for k even. Thus the model (29) becomes

$$\dot{u}_j = \frac{1}{h^2}\delta^2 u_j - \frac{1}{12h^2}\delta^4 u_j + \sigma\left[\phi_{j,0} - \tfrac{1}{24}\delta^2\phi_{j,0} + \tfrac{1}{12\sqrt{10}}\delta^2\hat{\phi}_j\right]. \quad (30)$$

   Instead of the infinite number of noise processes in (29), this model has only $2M$ noise modes for a spatial domain with M elements.

2. A further slight simplification of the model combines the two second difference δ^2 terms through replacing the noise components by the orthogonal combination

$$\begin{bmatrix}\psi_j(t)\\ \hat{\psi}_j(t)\end{bmatrix} = \begin{bmatrix}\sqrt{\frac{5}{7}} & -\sqrt{\frac{2}{7}}\\ \sqrt{\frac{2}{7}} & \sqrt{\frac{5}{7}}\end{bmatrix}\begin{bmatrix}\phi_j(t)\\ \hat{\phi}_j(t)\end{bmatrix}.$$

   Then the model (30) becomes

$$\dot{u}_j = \frac{1}{h^2}\delta^2 u_j - \frac{1}{12h^2}\delta^4 u_j + \sigma\left[\sqrt{\tfrac{5}{7}}\psi_j - \tfrac{1}{24}\sqrt{\tfrac{7}{5}}\delta^2\psi_j + \sqrt{\tfrac{2}{7}}\hat{\psi}_j\right]. \quad (31)$$

Although both these models have $2M$ noise modes, Appendix A proves that we cannot reduce this to the minimal M modes without making the model undesirably nonlocal. Centre manifold theory supports the particular weighted combination of noise in the model (31) as providing an appropriate balance between noise correlated between neighbouring elements, and independent noise in each element. Thus I commend models (30) or (31) as a discretisation for the stochastically forced diffusion equation (14).

3.5 Next nearest neighbour elements affect noise

Continue the analysis of the previous subsections to explore the terms in $\sigma\gamma^2$. These terms involve the forcing noise, coupling across five elements, and their interaction with the mechanism of diffusion.

Computer algebra [49, §5] uses iteration to compute the stochastic centre manifold model. The iteration, based upon the processes explained in previous subsections, terminates when the residuals of the forced diffusion equation (14) and the coupling IBCs (3) are zero to some specified order in the small parameters γ and σ. Then the Approximation Theorem for SPDEs [5, Theorem 8.1] assures us that the model is constructed to the same order of error. The main limitation of the computer algebra program [49] is that the infinite sums over Fourier modes must be truncated to finite sums. Extrapolating the patterns of coefficients from truncating to nine Fourier modes, the resultant improvement to (29) is the model

$$\begin{aligned} \dot{u}_j &= \frac{\gamma}{h^2}\delta^2 u_j - \frac{\gamma^2}{12h^2}\delta^4 u_j + \sigma\phi_{j,0} + \gamma\sigma\delta^2\left[-\frac{1}{24}\phi_{j,0} + \sum_{k=2,\text{even}}^{\infty}\frac{1}{\pi^2k^2}\phi_{j,k}\right] \\ &\quad + \gamma^2\sigma\delta^4\left[\frac{17}{2880}\phi_{j,0} - \sum_{k=2,\text{even}}^{\infty}\left(\frac{1}{12\pi^2k^2}+\frac{1}{\pi^4k^4}\right)\phi_{j,k}\right] \\ &\quad + \mathcal{O}\big(\sigma^3,\gamma^3\big)\,. \end{aligned} \tag{32}$$

Again the infinite sums of unknown noise terms may be simplified, but now through first replacing by two equivalent noise sources. A little Gramm–Schmidt orthonormalisation[1] shows

$$\begin{bmatrix} \sum_{k=2,\text{even}}^{\infty}\frac{1}{\pi^2k^2}\phi_{j,k} \\ \sum_{k=2,\text{even}}^{\infty}\left(\frac{1}{12\pi^2k^2}+\frac{1}{\pi^4k^4}\right)\phi_{j,k} \end{bmatrix} \equiv \begin{bmatrix} \frac{1}{12\sqrt{10}} & 0 \\ \frac{1}{112\sqrt{10}} & \frac{1}{5040\sqrt{2}} \end{bmatrix} \begin{bmatrix} \hat\phi_j(t) \\ \tilde\phi_j(t) \end{bmatrix},$$

for some new noises $\hat\phi_j(t)$ and $\tilde\phi_j(t)$ independent of $\phi_{j,0}$. Thus the model (32) is equivalent to

$$\begin{aligned} \dot{u}_j &= \frac{1}{h^2}\delta^2 u_j - \frac{1}{12h^2}\delta^4 u_j + \sigma\left[\phi_{j,0} - \frac{1}{24}\delta^2\phi_{j,0} + \frac{17}{2880}\delta^4\phi_{j,0}\right. \\ &\qquad\left. + \frac{1}{12\sqrt{10}}\delta^2\hat\phi_j - \frac{1}{112\sqrt{10}}\delta^4\hat\phi_j - \frac{1}{5040\sqrt{2}}\delta^4\tilde\phi_j\right], \end{aligned} \tag{33}$$

[1] The Gramm–Schmidt orthonormalisation uses the sums $\sum_{n=1}^{\infty} n^{-4} = \pi^4/90$, $\sum_{n=1}^{\infty} n^{-6} = \pi^6/945$ and $\sum_{n=1}^{\infty} n^{-8} = \pi^8/9450$.

upon also putting $\gamma = 1$ to recover a fully coupled model. It does not appear particularly useful to transform this model in order to reduce the number of differences δ appearing. See that the model (33) extends the lower order model (30) by including some fourth differences δ^4. These fourth differences of noise processes provides further subtle correlations among the noises and how they affect the diffusive dynamics in each element.

4 Nonlinear dynamics have irreducible noise interactions

The forced diffusion equation (14) is a linear SPDE; consequently, its models, such as (26), (32) and (33), are also linear in the noise magnitude σ. Using the example of the forced Burgers' SPDE (1), we now explore the discretisation of nonlinear SPDEs. Consider the iterative construction of the stochastic centre manifold model to effects quadratic in the magnitude σ of the noise. We primarily seek two types of terms in the model: terms in σ^2 as these generate mean drift forcing from the noise; and also terms in $\sigma^2 u_j$ as these reflect the influence of noise on the linear stability of Burgers' SPDE (1) [47, Figure 2] and [5, p.544].

4.1 Separate products of convolutions

In the iterative construction of a stochastic centre manifold we use the residuals of the governing SPDE to drive corrections, equation (15), to an approximate centre manifold model. In analysing nonlinear SPDEs, such as the stochastically forced Burgers' SPDE (1), products of memory convolutions appear in the residual. Furthermore, these convolutions will be over multiple time scales: even in the linear diffusion SPDE (14) the model (26) has double convolutions $\mathcal{Z}_{k,\ell}$. Seeking terms quadratic in the noise magnitude σ we thus generally have to deal with quadratic products of multiple convolutions appearing in the residual.

Obtain corrections from residuals To cater for the general case, define multiple convolutions. Let $\mathcal{Z}_{\mathbf{k}}$ denote the operator of multiple convolutions in time where vector $\mathbf{k}$ indicates the decay rate of the corresponding convolution, that is, the operator

$$\mathcal{Z}_{\mathbf{k}} = \mathcal{Z}_{k_1,k_2,\ldots} = \exp(-\beta_{k_1} t) \star \exp(-\beta_{k_2} t) \star \cdots \star \quad \text{and} \quad \mathcal{Z} = 1\,, \tag{34}$$

in terms of the convolution (20); consequently

$$\partial_t \mathcal{Z}_{k_1,k_2,\ldots} = -\beta_{k_1} \mathcal{Z}_{k_1,k_2,\ldots} + \mathcal{Z}_{k_2,\ldots}\,. \tag{35}$$

Note: the order of the convolutions does not matter [48, Appendix, e.g.]; however, keeping intact the order of the convolutions seems useful to most easily cancel like terms in the residual of the governing SPDE.

Recall that Sections 3.2–3.3 discusses how to determine updates u_j' and g_j' to the subgrid field and the model evolution when the residual contains terms linear in the noise: from (15) we may consider each term in the right-hand side in turn and solve equations of the form

$$\frac{\partial u_j'}{\partial t} - \frac{\partial^2 u_j'}{\partial x^2} + g_j' = \sigma f(\theta) \mathcal{Z}_{\mathbf{k}} \phi_{j,n}\,.$$

Sections 3.2–3.3 describe how to solve such equations. For nonlinear problems, such as the stochastically forced Burgers' equation (1), we additionally have to solve for corrections for each term of the form quadratic in the noise of the right-hand side of

$$\frac{\partial u_j'}{\partial t} - \frac{\partial^2 u_j'}{\partial x^2} + g_j' = \sigma^2 f(\theta) \mathcal{Z}_{\boldsymbol{\ell}} \phi_{j,n} \mathcal{Z}_{\mathbf{k}} \phi_{i,m}\,.$$

As in Sections 3.2–3.3, two cases arise:

- firstly, for each components of the subgrid structure $f(\theta)$ in $\operatorname{csn} p\theta$ for wavenumber $p \geq 1$, there is no difficulty in simply including in the correction to the subgrid field the component

  $$\sigma^2 \operatorname{csn} p\theta\, \mathcal{Z}_p \left[\mathcal{Z}_{\boldsymbol{\ell}} \phi_{j,n} \mathcal{Z}_{\mathbf{k}} \phi_{i,m}\right]$$

  with its extra convolution in time;
- secondly, for the component in $f(\theta)$ that is constant across an element, we have to separate the part of $\mathcal{Z}_{\boldsymbol{\ell}} \phi_{j,n} \mathcal{Z}_{\mathbf{k}} \phi_{i,m}$ that has a bounded integral in time, and hence updates the subgrid field u_j', from the so-called secular part that does not have a bounded integral and hence must update the model evolution g_j'.

Integrate by parts to separate To extract quadratic corrections to the evolution, use integration by parts so all non-integrable convolutions are reduced to the canonical form of the convolution being entirely over one of the noises in a quadratic term, that is, the canonical irreducible form is $\phi_{j,n}\mathcal{Z}_{\mathbf{k}}\phi_{i,m}$. Rewrite the convolution ODE (35) as

$$\beta_k \mathcal{Z}_{k.\mathbf{k}'} = -\partial_t \mathcal{Z}_{k.\mathbf{k}'} + \mathcal{Z}_{\mathbf{k}'} ,$$

where the vector of convolution parameters is $\mathbf{k} = k \cdot \mathbf{k}'$ so that k is the first component of $\mathbf{k}$, and $\mathbf{k}'$ is the vector (if any) of the second and subsequent components of $\mathbf{k}$. Then for any ϕ and ψ

$$\begin{aligned}
&\int \mathcal{Z}_{k\cdot\mathbf{k}'}\phi\mathcal{Z}_{\ell\cdot\boldsymbol{\ell}'}\psi\,dt \\
={}& \frac{1}{\beta_k+\beta_\ell}\int \beta_k\mathcal{Z}_{k\cdot\mathbf{k}'}\phi\mathcal{Z}_{\ell\cdot\boldsymbol{\ell}'}\psi + \mathcal{Z}_{k\cdot\mathbf{k}'}\phi\beta_\ell\mathcal{Z}_{\ell\cdot\boldsymbol{\ell}'}\psi\,dt \\
={}& \frac{1}{\beta_k+\beta_\ell}\int \left[-\partial_t\mathcal{Z}_{k\cdot\mathbf{k}'}\phi + \mathcal{Z}_{\mathbf{k}'}\phi\right]\mathcal{Z}_{\ell\cdot\boldsymbol{\ell}'}\psi + \mathcal{Z}_{k\cdot\mathbf{k}'}\phi\left[-\partial_t\mathcal{Z}_{\ell\cdot\boldsymbol{\ell}'}\psi + \mathcal{Z}_{\boldsymbol{\ell}'}\psi\right]\,dt \\
={}& \frac{1}{\beta_k+\beta_\ell}\int -\frac{\partial}{\partial t}\left[\mathcal{Z}_{k\cdot\mathbf{k}'}\phi\mathcal{Z}_{\ell\cdot\boldsymbol{\ell}'}\psi\right] + \mathcal{Z}_{\mathbf{k}'}\phi\mathcal{Z}_{\ell\cdot\boldsymbol{\ell}'}\psi + \mathcal{Z}_{k\cdot\mathbf{k}'}\phi\mathcal{Z}_{\boldsymbol{\ell}'}\psi\,dt \\
={}& -\frac{1}{\beta_k+\beta_\ell}\mathcal{Z}_{k\cdot\mathbf{k}'}\phi\mathcal{Z}_{\ell\cdot\boldsymbol{\ell}'}\psi + \frac{1}{\beta_k+\beta_\ell}\int \mathcal{Z}_{\mathbf{k}'}\phi\mathcal{Z}_{\ell\cdot\boldsymbol{\ell}'}\psi + \mathcal{Z}_{k\cdot\mathbf{k}'}\phi\mathcal{Z}_{\boldsymbol{\ell}'}\psi\,dt\,.
\end{aligned}$$

Observe that each of the two components in the integrand on the last line above has one fewer convolutions that the initial integrand. Thus this integration by parts in time t can be done until we reach terms of the form $\phi_{j,n}\mathcal{Z}_{\mathbf{k}}\phi_{i,m}$ in the integrand: assign all the terms that can be integrated to update the subgrid field u'_j. The irreducible terms remaining in the integrand, those in the form $\phi_{j,n}\mathcal{Z}_{\mathbf{k}}\phi_{i,m}$, must thus go to update the evolution g'_j.

Computer algebra [49, §6] implements these steps in its iteration to derive the asymptotic series of the stochastic centre manifold of an SPDE.

4.2 Strong model of stochastic dynamics are complex

Now look at the details of the discrete model of the stochastically forced Burgers' SPDE (1). Computer algebra [49] derives the following leading terms in the asymptotic series of the model $\dot{U}_j = g_j(\mathbf{U}, t, \epsilon)$. This model would be one of the most complicated models you ever wish to consider. The algebraic length reflects the complexity of the multiple physical processes acting on the subgrid scale structures forced by the rich stochastic spectrum of noise. Fortunately the arguments of the next section simplify the model significantly. Scan over the following model to the discussion following.

$$
\begin{aligned}
\dot{U}_j \;=\;& \gamma\frac{1}{h^2}\delta^2 U_j - \gamma^2\frac{1}{12h^2}\delta^4 U_j - \gamma\alpha\frac{1}{h}U_j\mu\delta U_j \\
&+ \sigma\Big\{ \left[1-\gamma\tfrac{1}{24}\delta^2+\gamma^2(\tfrac{3}{640}+\tfrac{1}{8\pi^4})\delta^4\right]\phi_{j,0} \\
&\quad + \left[\gamma\tfrac{1}{4\pi^2}\delta^2-\gamma^2(\tfrac{1}{48\pi^2}+\tfrac{1}{16\pi^4})\delta^4\right]\phi_{j,2} - \alpha\frac{2h}{\pi^2}U_j\phi_{j,1} \\
&\quad + \alpha\gamma\frac{1}{h^2\pi^2}\Big[U_j\left(\tfrac{8}{\pi^2}\mu\delta\phi_{j,0}-\tfrac{1}{4}\mu\delta\phi_{j,2}+(\tfrac{1}{12}+\tfrac{5}{3\pi^2})\delta^2\phi_{j,1}\right) \\
&\qquad + \mu\delta U_j\left(\tfrac14\phi_{j,2}+(\tfrac16+\tfrac{10}{3\pi^2})\mu\delta\phi_{j,1}\right) \\
&\qquad - \delta^2 U_j\left((\tfrac16+\tfrac{1}{3\pi^2})\phi_{j,1}-(\tfrac{1}{24}+\tfrac{5}{6\pi^2})\delta^2\phi_{j,1}\right)\Big] \\
&\quad - \alpha^2\frac{8h^2}{3\pi^4}U_j^2\phi_{j,0}\Big\} \\
&+ \sigma^2\Big\{\alpha\frac{h}{\pi^2}\left[-2\phi_{j,0}\mathcal{Z}_1\phi_{j,1}+\tfrac25\phi_{j,1}\mathcal{Z}_2\phi_{j,2}+\tfrac25\phi_{j,2}\mathcal{Z}_1\phi_{j,1}\right] \\
&\quad + \alpha\gamma\frac{1}{h\pi^2}\left(-32\phi_{j,0}\mathcal{Z}_{1,2}\mu\delta-\tfrac45\phi_{j,1}\mathcal{Z}_{2,2}\delta^2+\tfrac{32}{5}\phi_{j,2}\mathcal{Z}_{1,2}\mu\delta\right)\phi_{j,2} \\
&\quad + \alpha\gamma\frac{h}{\pi^2}\Big[\phi_{j,0}\Big(\tfrac{8}{\pi^2}\mathcal{Z}_1\mu\delta(\phi_{j,0}+\phi_{j,2})+(\tfrac{1}{12}+\tfrac{5}{3\pi^2})\mathcal{Z}_1\delta^2\phi_{j,1} \\
&\qquad\quad -(\tfrac14+\tfrac{8}{\pi^2})\mathcal{Z}_2\mu\delta\phi_{j,2}\Big)+\phi_{j,1}\mathcal{Z}_2\left(\tfrac15\delta^2\phi_{j,0}-(\tfrac{1}{20}+\tfrac{13}{150\pi^2})\phi_{j,2}\right) \\
&\qquad + \phi_{j,2}\Big(-\tfrac{8}{5\pi^2}\mathcal{Z}_1\mu\delta(\phi_{j,0}+\phi_{j,2})-(\tfrac{1}{60}+\tfrac{17}{75\pi^2})\mathcal{Z}_1\delta^2\phi_{j,1} \\
&\qquad\quad +(\tfrac18+\tfrac{4}{5\pi^2})\mathcal{Z}_2\mu\delta\phi_{j,2}\Big) \\
&\qquad + \delta^2\phi_{j,0}\,\mathcal{Z}_1\left(-(\tfrac{1}{12}+\tfrac{2}{15\pi^2})+(\tfrac{1}{24}+\tfrac{5}{6\pi^2})\delta^2\right)\phi_{j,1} \\
&\qquad - \delta^2\phi_{j,1}\,\mathcal{Z}_2\left((\tfrac{1}{60}+\tfrac{17}{75\pi^2})+(\tfrac{1}{120}+\tfrac{17}{150\pi^2})\delta^2\right)\phi_{j,2} \\
&\qquad - \delta^2\phi_{j,2}\,\mathcal{Z}_1\left((\tfrac{1}{20}+\tfrac{44}{75\pi^2})+(\tfrac{1}{120}+\tfrac{17}{150\pi^2})\delta^2\right)\phi_{j,1} \\
&\qquad + \mu\delta\phi_{j,0}\left((\tfrac16+\tfrac{10}{3\pi^2})\mathcal{Z}_1\mu\delta\phi_{j,1}+(\tfrac14-\tfrac{8}{5\pi^2})\mathcal{Z}_2\phi_{j,2}\right) \\
&\qquad - \mu\delta\phi_{j,1}(\tfrac{1}{30}+\tfrac{34}{75\pi^2})\mathcal{Z}_2\mu\delta\phi_{j,2} \\
&\qquad + \mu\delta\phi_{j,2}\left(-(\tfrac{1}{30}+\tfrac{34}{75\pi^2})\mathcal{Z}_1\mu\delta\phi_{j,1}+(\tfrac18-\tfrac{4}{5\pi^2})\mathcal{Z}_2\phi_{j,2}\right)\Big] \\
&\quad + \alpha^2\frac{1}{\pi^2}U_j\Big[-\tfrac{16}{3}\phi_{j,0}\left(2\mathcal{Z}_{1,2}+\tfrac{h^2}{\pi^2}\mathcal{Z}_2\right)\phi_{j,2} \\
&\qquad - \tfrac{8}{15}\phi_{j,1}\left(\mathcal{Z}_{2,1}-\tfrac{4h^2}{\pi^2}\mathcal{Z}_1\right)\phi_{j,1}+\tfrac{16}{15}\phi_{j,2}\left(2\mathcal{Z}_{1,2}+\tfrac{h^2}{\pi^2}\mathcal{Z}_2\right)\phi_{j,2}\Big]\Big\} \\
&+ \mathcal{O}\left(\sigma^3,\alpha^3+\gamma^3\right). \qquad (36)
\end{aligned}
$$

Table 1: number of terms in the evolution $\dot{U}_j = g_j(\mathbf{U}, t, \boldsymbol{\epsilon})$ when only three Fourier modes are used for the subgrid stochastic structures: the numbers in *italics* count the terms in the model (36). Expect many more terms when using more Fourier modes. Blank entries in the table are unknown.

σ^0

	γ^0	γ^1	γ^2	γ^3
α^3	0	0		
α^2	*0*	3	14	
α^1	*0*	*2*	8	19
α^0	*0*	*3*	*5*	7

σ^1

	γ^0	γ^1	γ^2	γ^3
α^3	1	13		
α^2	*1*	16	82	
α^1	*1*	*11*	45	93
α^0	*1*	*6*	*10*	14

σ^2

	γ^0	γ^1	γ^2	γ^3
α^3	9			
α^2	*6*	156		
α^1	*3*	*42*	238	
α^0	*0*	*0*	*0*	0

The model resolves noise, nonlinearity and inter-element interactions The model (36) is computed to residuals $\mathcal{O}\big(\sigma^3, \alpha^3 + \gamma^3\big)$ and hence the model has this order of error. The truncation to errors $\mathcal{O}\big(\sigma^3\big)$ ensures the model retains the interesting quadratic noise interaction terms parametrised by σ^2 seen in the last 14 lines of (36). The truncation to error $\mathcal{O}\big(\alpha^3+\gamma^3\big)$ resolves linear dynamics within and between next nearest neighbour elements, and nonlinear dynamics within and between nearest neighbour elements.

Truncate to three Fourier modes The model (36), complicated as it is, resolves just the first three Fourier modes of the forcing noise, namely $\phi = \sum_{k=0}^{2} \phi_{j,k}(t) \operatorname{csn} k\theta$. Similarly, the model (36) only resolves subgrid scale structure in the same three Fourier modes. In principle, the computer algebra [49] could generate models with many more subgrid scale modes; however, computer memory currently limits me to three modes for this analysis of the forced Burgers' SPDE (1).

Table 1 indicates the level of complexity of the multiparameter asymptotic series via a sort of Newton diagram. The table reports the number of terms in various parts of the model $\dot{U}_j = g_j(\mathbf{U}, t, \boldsymbol{\epsilon})$, there are vastly more terms describing the subgrid scale structure $u_j(\mathbf{U}, x, t, \boldsymbol{\epsilon})$ within each element. The critical point is that *if we pursue higher order truncations, or more Fourier modes, then the complexity of the model increases alarmingly.* Thus, for the moment, I choose to truncate the model as in (36).

Linear diffusion is a subset The diffusion model (32) appears in the first three lines of the nonlinear model (36) when the nonlinearity parameter $\alpha = 0$. The only differences are due to the finite truncation of the Fourier modes in this section: the infinite sums do not appear; and the coefficient of the $\gamma^2\sigma\delta^4\phi_{j,0}$ term has a small error from the modal truncation. The nonlinear model just modifies the model of linear diffusion.

Abandon fast time convolutions The undesirable feature of the large time model (36) is the inescapable appearance in the quadratic noise terms of fast time convolutions, such as $\mathcal{Z}_1\phi_{j,1} = \exp(-\pi^2 t/h^2)\star\phi_1$ and $\mathcal{Z}_{1,2}\phi_{j,2} = \exp(-\pi^2 t/h^2)\star\exp(-4\pi^2 t/h^2)\star\phi_{j,2}$. These require resolution of the subgrid fast time scales in order to maintain fidelity with the original Burgers' SPDE (1) and so require incongruously small time steps for a supposedly slowly evolving model. However, maintaining fidelity with the details of the white noise source $\phi(x,t)$ is a pyrrhic victory when we are only interested in the relatively slow long term dynamics. Instead we need only those parts of the quadratic noise factors, such as $\phi_{j,0}\mathcal{Z}_1\phi_{j,1}$ and $\phi_{j,0}\mathcal{Z}_{1,2}\phi_{j,2}$, that *over the macroscopic time scales* are firstly correlated with the other processes that appear and secondly independent of the other processes. The next section explores how these components of the quadratic noises not only introduce factors of effectively *new independent* noises into the model, but also introduce a deterministic drift due to stochastic resonance (as also noted by Drolet & Vinal [16]).

5 Stochastic resonance influences deterministic dynamics

Chao & Roberts [12, 47, 48] argued that quadratic terms involving memory integrals of the noise were effectively new drift and new noise terms when viewed over the long time scales of the relatively slow evolution of a model such as (36). The arguments of Chao & Roberts [12, 47, 48] rely upon the noise being stochastic white noise. The strong model (36) faithfully tracks any given realisation of the original Burgers' SPDE [5, Theorem 7.1(i), e.g.] whether the forcing is deterministic or stochastic; however, now we derive a weak model for the case of stochastic forcing. The weak model only maintains fidelity to solutions of the original Burgers' SPDE (1) in a weak sense—we cannot know which realisations ensure a match between the model and Burgers' SPDE because of the effectively new noises on the macroscale of the model.

Analogously, Just et al. [24] argued that fast time deterministic chaos appears as noise when viewed over long time scales. In the case of deterministic forcing of Burgers' equation (1), perhaps similar arguments to those of Just et al. could map the strong model (36), with its troublesome fast time scales, into a stochastic model over the large time scales of interest.

5.1 Canonical quadratic noise interactions

In the strong model (36) we need to understand and summarise the long term effects of the quadratic noises that appear in the form $\phi_j \mathcal{Z}_k \phi_i$ and $\phi_j \mathcal{Z}_{k,\ell} \phi_i$, where here ϕ_i and ϕ_j represent the various possibilities for the components $\phi_{j,k}$. The noises ϕ_i and ϕ_j may be independent or they may be the same process depending upon the term under consideration. We aim to replace such noise terms by a corresponding stochastic differential dy/dt for some Stratonovich stochastic process y with some drift and volatility: $dy = ()dt + ()dW$ for a Wiener process W. Thus we must understand the long term dynamics of Stratonovich stochastic processes y_1 and y_2 defined via the nonlinear SDEs

$$\frac{dy_1}{dt} = \phi_j \mathcal{Z}_k \phi_i \quad \text{and} \quad \frac{dy_2}{dt} = \phi_j \mathcal{Z}_{\ell,k} \phi_i \,. \tag{37}$$

Use the argument in [48] to proceed. Name the two convolutions that appear in the nonlinear terms (37) as $z_1 = \mathcal{Z}_k \phi_i$ and $z_2 = \mathcal{Z}_{\ell,k} \phi_i$. They satisfy the SDEs (25). Now put the SDEs (37) and (25) together: we must understand the long term properties of y_1 and y_2 governed by the coupled system

$$\begin{aligned} \dot y_1 &= z_1 \phi_j \,, & \dot z_1 &= -\beta_k z_1 + \phi_i \,, \\ \dot y_2 &= z_2 \phi_j \,, & \dot z_2 &= -\beta_\ell z_2 + z_1 \,. \end{aligned} \tag{38}$$

There are two cases labelled by s to consider: when $i = j$ the two source noises ϕ_i and ϕ_j are identical, $s = 1$; but when $i \neq j$ the two noise sources are independent, $s = 0$.

Centre manifold theory applied to the Fokker–Planck equation for the system (38) proves [48, §4] that as time $t \to \infty$ the probability density function (PDF)

$$\begin{aligned} P(\mathbf{y}, \mathbf{z}, t) \quad \to \quad & A \exp\left\{-(\beta_k + \beta_\ell) \left[z_1^2 - 2\beta_\ell z_1 z_2 + \beta_\ell(\beta_k + \beta_\ell) z_2^2\right]\right\} \\ & \times \left\{ p - s \left[z_1^2 - 2\beta_\ell z_1 z_2 + 2\beta_\ell(\beta_k + \beta_\ell) z_2^2 + B_1\right] \frac{\partial p}{\partial y_1} \right. \end{aligned}$$

$$- s\left[(\beta_k+\beta_\ell)z_2^2+B_2\right]\frac{\partial p}{\partial y_2}+\cdots\Bigg\}\,, \tag{39}$$

exponentially quickly for some normalisation constants A, B_1 and B_2, and for some evolving $p(\mathbf{y},t)$. Interpret $p(\mathbf{y},t)$ as a type of conditional probability density. Simultaneously with finding the next order corrections to this PDF $P(\mathbf{y},\mathbf{z},t)$, I found in [48] that the relatively slowly varying, quasi-conditional probability density p evolves according to the Kramers–Moyal expansion [34, 33, 56, e.g.]

$$\frac{\partial p}{\partial t}=-\tfrac{1}{2}s\frac{\partial p}{\partial y_1}+\frac{1}{4\beta_k}\begin{bmatrix}1 & \frac{1}{\beta_k+\beta_\ell}\\ \frac{1}{\beta_k+\beta_\ell} & \frac{1}{\beta_\ell(\beta_k+\beta_\ell)}\end{bmatrix}:\boldsymbol{\nabla\nabla} p+\cdots\,. \tag{40}$$

The centre manifold relevance theorem [46, §2.2.2, e.g.] assures us that this PDE for the evolution of the quasi-PDF $p(\mathbf{y},t)$ models the dynamics for large time. The PDE describes the evolution of the interesting modes $\mathbf{y}$ that arise as noises interacting nonlinearly with each other.

5.2 Translate to a corresponding SDE

Interpret (40) as a Fokker–Planck equation for the SDEs

$$\dot y_1=\tfrac{1}{2}s+\frac{\psi_1(t)}{\sqrt{2\beta_k}}\quad\text{and}\quad \dot y_2=\frac{1}{\beta_k+\beta_\ell}\left(\frac{\psi_1(t)}{\sqrt{2\beta_k}}+\frac{\psi_2(t)}{\sqrt{2\beta_\ell}}\right)\,. \tag{41}$$

Of course there are many coupled SDEs whose Fokker–Planck equation is (40): for example, Just et al. [24] choose their volatility matrix to be the positive definite, symmetric square root of the diffusivity matrix in (40). For our purposes any of the possible volatility matrices would suffice: in constructing a weak model via the Fokker–Planck equations we necessarily lose fidelity of paths, and now only require fidelity of distributions and correlations. To obtain the form of the noise terms in the SDE (41) use the unique Cholesky factorisation of the symmetric diffusion matrix in (40). Using the Cholesky decomposition to determine the volatility matrix in the SDE (41) ensures that nearly half the terms in the volatility matrix are zero. Furthermore, using the Cholesky decomposition also ensures that if we were to analyse the dynamics of Burgers' SPDE (1) to higher orders, then the higher order convolutions of noise that would arise do not change this 2×2 Cholesky factorisation [48, §5]. Thus the SDEs (41) are our a model for the evolution of the irreducible convolutions (37) over long times.

As argued by Chao & Roberts [12, 47] and proved in [48, Appendix], the two $\psi_i(t)$ are new noises independent of ϕ_i and ϕ_j *over long time scales*. The remarkable feature to see in the SDEs (41) is that for the case of identical noise, $\phi_i = \phi_j$, that is $s = 1$, there is a mean drift $\frac{1}{2}$ in the stochastic process y_1; there is no mean drift in any other process nor in the other case, $s = 0$.

You might query the role of the neglected terms, indicated by the ellipsis $\cdots$, in the Kramers–Moyal expansions of the PDF (39) and the proposed Fokker–Planck equation (40). In the PDF (39) the neglected terms just provide more details of the non-Gaussian structure of the PDF in the slowly evolving long time dynamics. The effects of the neglected terms in (40) correspond to algebraically decaying departures from the second order truncation that we interpret as a Fokker–Planck equation. Such algebraically decaying transients may represent slow decay of non-Markovian effects among the $\mathbf{y}$ variables. However, the truncation (40) that we interpret as a Fokker–Planck equation is the lowest order *structurally stable* model and so will adequately model the dynamics of y_1 and y_2 over the longest time scales. Just et al. [24] in their equation (11) similarly truncate to second order.

5.3 Transform the strong model (36) to be usefully weak.

The quadratic noises in (36) involve the convolutions $\mathcal{Z}_1$ and $\mathcal{Z}_2$ which have respective decay rates $\beta_1 = \pi^2/h^2$ and $\beta_2 = 4\pi^2/h^2$. Thus via various instances of the SDEs (41), to obtain a model for long time scales we replace the quadratic noises in (36) as follows:

$$\begin{aligned}
\phi_{j,n}\mathcal{Z}_k\phi_{i,m} &\mapsto \frac{1}{2}\delta_{ij}\delta_{mn} + \frac{h}{k\pi\sqrt{2}}\psi_{nmk}(t)\,, \\
\phi_{j,n}\mathcal{Z}_{k,\ell}\phi_{i,m} &\mapsto \frac{h^3}{\pi^3(k^2+\ell^2)}\left[\frac{1}{\ell\sqrt{2}}\psi_{nm\ell}(t) + \frac{1}{k\sqrt{2}}\psi_{nm\ell k}(t)\right]\,, \qquad (42)
\end{aligned}$$

where ψ_{nmk} and $\psi_{nm\ell k}$ are the effectively new and *independent* white noises, that is, derivatives of new independent Wiener processes. I omit the subscripts of i and j on ψ and henceforth on ϕ because they are redundant when we record the model using centred mean and difference operators. Note that δ_{ij} and δ_{mn}, with its pair of subscripts, do not denote a centred difference but rather denote the Dirac delta to cater for the case of the self interaction of a noise when there is a mean drift effect.

Computer algebra [49, §9] implements the transformations (42) in the strong model (36) to derive the weak model

$$
\begin{aligned}
\dot{U}_j = {} & \gamma\frac{1}{h^2}\delta^2 U_j - \gamma^2\frac{1}{12h^2}\delta^4 U_j - \gamma\alpha\frac{1}{h}U_j\mu\delta U_j \\
& + \sigma\left[\phi_0 - .2026\alpha h U_j\phi_1 - .02738\alpha^2h^2U_j^2\phi_2\right] \\
& + \gamma\sigma\delta^2\left(-.04167\phi_0 + .02533\phi_2\right) + \gamma^2\sigma\delta^4\left(.005971\phi_0 - .002752\phi_2\right) \\
& + \alpha h\gamma\sigma\{\mu\delta U_j\left[.02533\phi_2 + .05111\mu\delta\phi_1\right] \\
& \quad + \delta^2 U_j\left[-.02031\phi_1 + .01278\delta^2\phi_1\right] \\
& \quad + U_j\left[\mu\delta\left(.08213\phi_0 - .02533\phi_2\right) + .02555\delta^2\phi_1\right]\} \\
& + \alpha h^2\sigma^2\left(-.04561\psi_{011} + .004561\psi_{122} + .009122\psi_{211}\right) \\
& + \alpha h^2\gamma\sigma^2[\mu\delta_2(.01849\psi_{001} + .01849\psi_{021} - .01479\psi_{0212} - .01949\psi_{022} \\
& \qquad - .003697\psi_{201} - .003697\psi_{221} + .002958\psi_{2212} + .003828\psi_{222}) \\
& \quad + \mu\delta_1\left(.001002\psi_{022} + .0005011\psi_{222}\right) \\
& \quad + \mu\delta_1\mu\delta_2\left(.0115\psi_{011} - .0009038\psi_{122} - .001808\psi_{211}\right) \\
& \quad + \delta_2^2(.005752\psi_{011} + .0002311\psi_{102} - .0007858\psi_{122} \\
& \qquad - .0001155\psi_{1222} - .0009038\psi_{211}) \\
& \quad + \delta_1^2\left(-.002209\psi_{011} - .0004519\psi_{122} - .002496\psi_{211}\right) \\
& \quad + \delta_1^2\delta_2^2\left(.002876\psi_{011} - .000226\psi_{122} - .0004519\psi_{211}\right)] \\
& + \alpha^2h^3\sigma^2U_j(-.004929\psi_{0212} - .008626\psi_{022} + .004929\psi_{111} \\
& \quad - .0002465\psi_{1112} - .0001232\psi_{112} + .0009859\psi_{2212} + .001725\psi_{222}) \\
& + .01643\alpha^2h^2\sigma^2U_j + \mathcal{O}\left(\sigma^3, \alpha^3 + \gamma^3\right), \qquad (43)
\end{aligned}
$$

where $\mu\delta_1$ and δ_1^2 denote differences in the first grid variable implicit in the noises ψ, whereas $\mu\delta_2$ and δ_2^2 denote differences in the second implicit grid variable. The value of the model (43) is that there are no fast time scale processes within it. It is truly a model of the long time dynamics. Of course the fluctuating processes ϕ_n and ψ_{nmk} have fluctuations over all time scales, but we know how to integrate these with macroscopic time steps. The complexity of the model (43) reflects the complexity of the inter-element interactions and the subgrid scale processes resolved in this rigorous approach to forming discretisations of SPDEs.

The stochastic components of the model (43), and (36), are actually more complex than it might appear: the difference operatorss hide a lot of detail. Any one apparent noise source ψ_{nmk} actually represents $5M$ independent noise sources over all the M elements. In order to clarify all the discrete differences of ψ_{nmk} that appear, temporarily reinstate the implicit subscripts.

For the jth element we need the nine noise components ψ_{jjnmk}, $\psi_{j\pm1,jnmk}$, $\psi_{j,j\pm1,nmk}$ and $\psi_{j\pm1,j\pm1,nmk}$ in order to compute all the differences that appear in (43). Of these, seven of the noises are used in computing the differences in the $(j\pm1)$th element, and two are also used in computing the differences in the $(j \pm 2)$th element. Consequently each of the $\sigma^2\gamma$ noises that appear in (43) actually represent, in nett effect, five independent noise sources for each element. Such terms reflect subtle cross-correlations between the stochastic dynamics within neighbouring finite elements.

Stochastic induced drift affects stability The terms quadratic in the noise magnitude, indicated by σ^2, are particularly complicated; with relatively small numerical coefficient perhaps we could ignore them. Except one important term. The last known term in (43), namely $+.01643\alpha^2h^2\sigma^2U_j$, is a mean effect of the noise interacting through the nonlinearity. The positive coefficient of this term shows that the self interactions of each of the many subgrid scale noises actually act to promote growth of macroscale structures in the Burgers' dynamics. For many practical purposes we could probably ignore all the σ^2 terms except this one term because of its potential macroscopic effects over long times. Indeed, because of its potential importance, I included this term in the introductory model (4).

Similarly, Boxler [5, p.544], Drolet & Vinals [16, 17] and Knobloch & Weisenfeld [27] and Vanden–Eijnden [57, p.68] found stability modifying linear terms in their analyses of stochastically perturbed bifurcations and systems. Recall that the Relevance Theorem by Boxler [5, Theorem 7.3(a)] proves that the stability of an original SDE is the same as the model SDE on the stochastic centre manifold; thus growth promoting terms in the model SDEs do represent dynamics of the Burgers' SPDE (1). Analogously to these effects of microscale noise on the macroscale dynamics, Just et al. [24] sought to determine how microtime deterministic chaos, not noise, translates into a new effective stochastic noise in the slow modes of a deterministic dynamical system. The analysis here shows that noise in many subgrid modes contribute to destabilise the trivial equilibrium $u = 0$.

5.4 Consolidate the new noise

Orthonormalisation simplifies the representation of the effects of all the noise terms in (43): this subsection reduces the 16 quadratic noises to just seven equivalent noise sources. Because the noise terms appearing in (43) are unknown in detail, we may replace linear combinations of them by one equivalent noise term as Sections 3.4 and 3.5 did for the model of linear diffusion. Recall that we have to be careful to maintain the correct correlations between the various places that the noise terms appear. The situation is fiendishly more complicated in the highly complex model (43) for the nonlinear Burgers' SPDE (1), in comparison to the earlier diffusion SPDE (14), because the noises appear in many more places in different combinations (indicated by the parenthetical groupings in (43)).

- Because of the severe truncation in the number of retained Fourier modes, there is no significant simplification possible in the terms linear in the noise magnitude σ: the ϕ_n noises just occur in too many places; if we had an infinite sum of Fourier modes, as in Section 3.4 and 3.5, then the infinite noise components perhaps would be reduced to the form of (43).
- Now turn to the quadratic noise terms in (43). Computer algebra [49, §9] extracts the eight different combinations of noises ψ in the weak model (43). Then a Gramm–Schmidt orthonormalisation of the vectors of coefficients is essentially a QR decomposition of the transpose of the matrix of noise coefficients: namely, factor the noise contributions to $R^T Q^T \boldsymbol{\psi}$ where $\boldsymbol{\psi}$ is the vector of noise processes, Q^T is an orthogonal matrix, and R^T is a lower triangular matrix. Then $\boldsymbol{\chi} = Q^T\boldsymbol{\psi}$ are a vector of new independent noise processes to replace $\boldsymbol{\psi}$. For our weak model (43), only the first seven rows of R^T are non-zero, and hence only the first seven components of the new noises $\boldsymbol{\chi}$ are significant. Thus the seven new noises $\boldsymbol{\chi}$ replace the 16 noises $\boldsymbol{\psi}$ with coefficients in R^T.

Computer algebra [49, §9] computes the QR factorisation of the quadratic noise coefficients in the weak model (43). However, the computer algebra also handles the case of four Fourier modes, instead of the three Fourier modes used to compute (43). There is a significant difference in the amount of detail: with truncation to four Fourier modes the weak model (43) has 92 terms in its centred difference form; in comparison, with truncation to three Fourier modes the weak model (43) has 53 terms in its centred difference form. However, upon replacing the quadratic noises $\boldsymbol{\psi}$ to equivalent noises $\boldsymbol{\chi}$ the resultant weak model has complexity largely independent of the number of retained Fourier modes. The model from four Fourier modes is

$$\dot{U}_j = \gamma\frac{1}{h^2}\delta^2 U_j - \gamma^2\frac{1}{12h^2}\delta^4 U_j - \gamma\alpha\frac{1}{h}U_j\mu\delta U_j$$

$$
\begin{aligned}
&+\sigma\left[\phi_0+\alpha h U_j(-.2026\phi_1+.02252\phi_3)-.02555\alpha^2h^2U_j^2\phi_2\right]\\
&+\gamma\sigma\delta^2(-.04167\phi_0+.02533\phi_2)+\gamma^2\sigma\delta^4(.005971\phi_0-.002752\phi_2)\\
&+\alpha h\gamma\sigma\{\mu\delta U_j[.02533\phi_2+\mu\delta(.05111\phi_1-.00649\phi_3)]\\
&\quad+\delta^2U_j\left[(-.02031\phi_1-.0001769\phi_3)+\delta^2(.01278\phi_1-.001622\phi_3)\right]\\
&\quad+U_j\left[\mu\delta(.08314\phi_0-.02533\phi_2)+\delta^2(.02555\phi_1-.003245\phi_3)\right]\big\}\\
&+.04681\alpha h^2\sigma^2\chi_1\\
&+\alpha h^2\gamma\sigma^2[\mu\delta_2(.02163\chi_2+.02949\chi_3)\\
&\quad+\mu\delta_1(-.0006027\chi_2-.000111\chi_3+.0008305\chi_4)\\
&\quad+\mu\delta_1\mu\delta_2(-.01168\chi_1+.000587\chi_5)\\
&\quad+\delta_2^2(-.005875\chi_1+.0001334\chi_5+.0004103\chi_6)\\
&\quad+\delta_1^2(.001608\chi_1-.002696\chi_5-.0005192\chi_6+.001116\chi_7)\\
&\quad+\delta_1^2\delta_2^2(-.00292\chi_1+.0001468\chi_5)]\\
&+.01126\alpha^2h^3\sigma^2U_j\chi_2+.01751\alpha^2h^2\sigma^2U_j+\mathcal{O}\big(\sigma^3,\alpha^3+\gamma^3\big)\,. \qquad (44)
\end{aligned}
$$

In this weak model, the new noises χ_n implicitly have two subscripts to parametrise noise in pairs of nearby elements, as for ψ_{nmk}: these reflect some of the subtle correlations between neighbouring elements. The differences between this and the previous weak model (43) are the following:

- it is derived by resolving four subgrid Fourier modes within each element instead of three;
- the effects of the subgrid scale noise ϕ_3 explicitly appear;
- the implicit effects of the subgrid scale noise ϕ_3 change some of the coefficients slightly—for example, the interesting mean destabilising term $.01643\alpha^2h^2\sigma^2U_j$ in (43) is more accurately $.01751\alpha^2h^2\sigma^2U_j$ in (44);
- the multitude of nonlinearity induced quadratic noise interactions have been replaced by just seven completely equivalent noise processes χ (although, as discussed earlier for ψ, these seven implicitly represent five times as many independent noise processes per element!).

6 Conclusion

The crucial virtue of the weak model (44), as also recognised by Just et al. [24], is that we may accurately take large time steps as *all* the fast dynamics have been eliminated. The critical innovation here is we have demonstrated, via the particular example of Burgers' SPDE (1), how it is feasible to analyse the net effect of many independent subgrid stochastic effects both within an element and between neighbouring elements. Observe that we remove all memory integrals (convolutions) from the model and that quadratic effects in the noise processes effectively generate a mean drift and abstract effectively new noises. General formulae for modelling quadratic noise [48] together with the iterative construction of stochastic centre manifold models [40] empower us to model a wide range of SPDEs.

Theoretical support for the models comes from dividing the spatial domain into finite sized elements with coupling conditions (3), invoking stochastic centre manifold theory [5], and then systematically analysing the subgrid processes together with the appropriate physical coupling between the elements. This approach builds on its success in discretely modelling deterministic PDEs [42, 44, 29, e.g.].

We sought solutions to Burgers' SPDE (1) on a domain of size L with periodic boundary conditions so that the discrete models are homogeneous. What about other domains with physical boundary conditions at their extremes? The coupling parameter γ controls the information flow between adjacent elements; thus our truncation to a finite power in γ restricts the influence in the model of any physical boundaries to just a few elements near that physical boundary. Crucially, the approach proposed here is based purely upon the *local* dynamics on small elements while seeking to maintain fidelity with the solutions of the original SPDE. In the interior, the methods described here remain unchanged and thus produce the same models. The same methodology, but with different details can account for physical boundaries to produce a discrete model valid across the whole domain. This has already been shown for the deterministic Burgers' PDE [45] and shear dispersion in a channel [29].

Future research may find a useful simplification of the analysis used here if it can determine the mean drift terms, quadratic in σ^2, without having to compute the other σ^2 terms.

This approach to spatial discretisation may be extended easily to higher spatial dimensions as already commenced for deterministic PDEs [29, 30]. Because of the need to decompose the stochastic residuals into eigenmodes on each element, the application to higher spatial dimensions are likely to require tessellating space into simple rectangular elements for SPDEs.

A Local simple noise is impossible

I prove that we cannot reduce the number of the noise terms in (31) without making the noises non-local. Suppose we have a noise term forcing the model of the form

$$\begin{bmatrix} a + b\delta^2 & 1 \end{bmatrix} \begin{bmatrix} \psi \\ \hat\psi \end{bmatrix} .$$

Seek to simplify by an orthogonal transformation to new noises $(\chi, \hat\chi)$ by

$$\begin{bmatrix} \psi \\ \hat\psi \end{bmatrix} = \begin{bmatrix} Q_1 & Q_3 \\ Q_2 & Q_4 \end{bmatrix} \begin{bmatrix} \chi \\ \hat\chi \end{bmatrix} .$$

Then the noise term

$$\begin{aligned} \begin{bmatrix} a + b\delta^2 & 1 \end{bmatrix} \begin{bmatrix} \psi \\ \hat\psi \end{bmatrix} &= \begin{bmatrix} a + b\delta^2 & 1 \end{bmatrix} \begin{bmatrix} Q_1 & Q_3 \\ Q_2 & Q_4 \end{bmatrix} \begin{bmatrix} \chi \\ \hat\chi \end{bmatrix} \\ &= \begin{bmatrix} (a + b\delta^2)Q_1 + Q_2 & (a + b\delta^2)Q_3 + Q_4 \end{bmatrix} \begin{bmatrix} \chi \\ \hat\chi \end{bmatrix} . \end{aligned}$$

The noise $\hat\chi$ is eliminated and the model made simpler only when $(a + b\delta^2)Q_3 + Q_4 = 0$; that is,

$$Q_4 = -(a + b\delta^2)Q_3 .$$

But the entire Q matrix must also be orthogonal. Hence, among other equations,

$$Q_1Q_2^\dagger + Q_3Q_4^\dagger = 0 \quad \text{and} \quad Q_2Q_2^\dagger + Q_4Q_4^\dagger = 1 .$$

Using Q_4 then makes these

$$Q_1Q_2^\dagger = (a + b\delta^2)Q_3Q_3^\dagger \quad \text{and} \quad Q_2Q_2^\dagger + (a + b\delta^2)^2Q_3Q_3^\dagger = 1 ,$$

recognising that throughout all operator products are commutative as, in a spatially homogeneous domain, the operators are identical at each point in the spatial domain and hence the eigenvectors of the operators are identical, namely spatial Fourier modes. Use the first equation to eliminate $Q_3Q_3^\dagger$ from the second to obtain

$$\begin{aligned} & Q_2Q_2^\dagger + (a + b\delta^2)Q_1Q_2^\dagger = 1 \\ \Leftrightarrow \quad & \left[Q_2 + (a + b\delta^2)Q_1\right] Q_2^\dagger = 1 . \end{aligned}$$

But if Q_1 and Q_2 are to be *local* operators, then this equation can only be satisfied if both $Q_2+(a+b\delta^2)Q_1$ and $Q_2^\dagger$ are scalars. Hence $(a+b\delta^2)Q_1$ must be a scalar which is impossible for a *local* operator Q_1. This contradiction proves we cannot reduce the number of noise modes in (31) while maintaining locality in the model.

I expect analogous results for more complex models, and so, for example, do not even consider simplifying the analogous noise components of (33).

References

[1] L. Arnold, N. Sri Namachchivaya, and K. R. Schenk-Hoppé. Toward an understanding of stochastic Hopf bifurcation: a case study. *Intl. J. Bifurcation & Chaos*, 6:1947–1975, 1996.

[2] A. Bensoussan and F. Flandoli. Stochastic inertial manifolds. *Stochastics and Stochastics Rep.*, 53:13–39, 1995.

[3] Nils Berglund and Barbara Gentz. Geometric singular perturbation theory for stochastic differential equations. Technical report, [http://arXiv.org/abs/math.PR/0204008], 2003.

[4] D. Blomker, M. Hairer, and G. A. Pavliotis. Modulation equations: stochastic bifurcation in large domains. Technical report, [http://arXiv.org/abs/math-ph/040801], 2004.

[5] P. Boxler. A stochastic version of the centre manifold theorem. *Probab. Th. Rel. Fields*, 83:509–545, 1989.

[6] P. Boxler. How to construct stochastic center manifolds on the level of vector fields. *Lect. Notes in Maths*, 1486:141–158, 1991.

[7] Tomas Caraballo, Jose A. Langa, and James C. Robinson. A stochastic pitchfork bifurcation in a reactiondiffusion equation. *Proc. R. Soc. Lond. A*, 457:2041–2061, 2001.

[8] J. Carr. *Applications of centre manifold theory*, volume 35 of *Applied Math. Sci.* Springer–Verlag, 1981.

[9] J. Carr and R. G. Muncaster. The application of centre manifold theory to amplitude expansions. I. ordinary differential equations. *J. Diff. Eqns.*, 50:260–279, 1983.

[10] J. Carr and R. G. Muncaster. The application of centre manifold theory to amplitude expansions. II. Infinite dimensional problems. *J. Diff. Eqns*, 50:280–288, 1983.

[11] H. C. Chang. Onset of nonlinear waves on falling films. *Phys. Fluids A*, 1:1314–1327, 1989.

[12] Xu Chao and A. J. Roberts. On the low-dimensional modelling of Stratonovich stochastic differential equations. *Physica A*, 225:62–80, 1996.

[13] P. H. Coullet, C. Elphick, and E. Tirapegui. Normal form of a Hopf bifurcation with noise. *Physics Letts*, 111A(6):277–282, 1985.

[14] S. M. Cox and A. J. Roberts. Initial conditions for models of dynamical systems. *Physica D*, 85:126–141, 1995.

[15] J. Dolbow, M. A. Khaleel, and J. Mitchell. Multiscale mathematics initiative: A roadmap. report from the 3rd DoE workshop on multiscale mathematics. Technical report, Department of Energy, USA, `http://www.sc.doe.gov/ascr/mics/amr`, December 2004.

[16] Francois Drolet and Jorge Vinals. Adiabatic reduction near a bifurcationn in stochastically modulated systems. *Phys. Rev. E*, 57:5036–5043, 1998. [`http://link.aps.org/abstract/PRE/v57/p5036`].

[17] Francois Drolet and Jorge Vinals. Adiabatic elimination and reduced probability distribution functions in spatially extended systems with a fluctuating control parameter. *Phys. Rev. E*, 64:026120, 2001. [`http://link.aps.org/abstract/PRE/v64/e026120`].

[18] Jinqiao Duan, Kening Lu, and Bjorn Schmalfuss. Invariant manifolds for stochastic partial differential equations. *The Annals of Probability*, 31:2109–2135, 2003.

[19] C. Foias, M. S. Jolly, I. G. Kevrekidis, and E. S. Titi. Dissipativity of numerical schemes. *Nonlinearity*, 4:591–613, 1991.

[20] C. Foias and E. S. Titi. Determining nodes, finite difference schemes and inertial manifolds. *Nonlinearity*, 4:135–153, 1991.

[21] J. Fujimura. Methods of centre manifold multiple scales in the theory of nonlinear stability for fluid motions. *Proc Roy Soc Lond A*, 434:719–733, 1991.

[22] C. W. Gear, Ju Li, and I. G. Kevrekidis. The gap-tooth method in particle simulations. *Phys. Lett. A*, 316:190–195, 2003.

[23] W. Grecksch and P. E. Kloeden. Time-discretised Galerkin approximations of parabolic stochastics PDEs. *Bull. Austral Math. Soc.*, 54:79–85, 1996.

[24] Wolfram Just, Holger Kantz, Christian Rodenbeck, and Mario Helm. Stochastic modelling: replacing fast degrees of freedom by noise. *J. Phys. A: Math. Gen.*, 34:3199–3213, 2001.

[25] Yuri Kabanov and Sergei Pergamenshchikov. *Two-scale stochastic systems*, volume 49 of *Applications of mathematics: stochastic modelling and applied probability*. Springer, 2003.

[26] P. E. Kloeden and E. Platen. *Numerical solution of stochastic differential equations*, volume 23 of *Applications of Mathematics*. Springer–Verlag, 1992.

[27] E. Knobloch and K. A. Wiesenfeld. Bifurcations in fluctuating systems: The centre manifold approach. *J. Stat Phys*, 33:611–637, 1983.

[28] T. Mackenzie and A. J. Roberts. Holistic finite differences accurately model the dynamics of the Kuramoto–Sivashinsky equation. *ANZIAM J.*, 42(E):C918–C935, 2000. `http://anziamj.austms.org.au/V42/CTAC99/Mack`.

[29] T. MacKenzie and A. J. Roberts. Holistic discretisation of shear dispersion in a two-dimensional channel. In K. Burrage and Roger B. Sidje, editors, *Proc. of 10th Computational Techniques and Applications Conference CTAC-2001*, volume 44, pages C512–C530, March 2003. `http://anziamj.austms.org.au/V44/CTAC2001/Mack`.

[30] Tony MacKenzie. *Create accurate numerical models of complex spatio-temporal dynamical systems with holistic discretisation*. PhD thesis, University of Southern Queensland, 2005.

[31] G. N. Mercer and A. J. Roberts. A complete model of shear dispersion in pipes. *Jap. J. Indust. Appl. Math.*, 11:499–521, 1994.

[32] E. Meron and I. Procaccia. Theory of chaos in surface waves: The reduction from hydrodynamics to few-dimensional dynamics. *Phys. Rev. Lett.*, 56:1323–1326, 1986.

[33] R. Metzler. Non-homogeneous random walks, generalised master equations, fractional Fokker–Planck equations, and the generalised Kramers–Moyal expansion. *Eur. Phys. J. B*, 19:249–258, 2001.

[34] A. Naert, R. Friedrich, and J. Peinke. Fokker–Planck equation for the energy cascade in turbulence. *Physical Rev. E*, 56:6719–6722, 1997.

[35] N. Sri Namachchivaya and Y. K. Lin. Method of stochastic normal forms. *Int. J. Nonlinear Mechanics*, 26:931–943, 1991.

[36] National Physical Laboratory. *Modern Computing Methods*, volume 16 of *Notes on Applied Science*. Her Majesty's Stationary Office, 1961.

[37] I. Procaccia. Universal properties of dynamically complex systems: the organisation of chaos. *Nature*, 333:618–623, 1988. 16th June.

[38] A. J. Roberts. Low-dimensional models of thin film fluid dynamics. *Phys. Letts. A*, 212:63–72, 1996.

[39] A. J. Roberts. Low-dimensional modelling of dynamical systems. Technical report, `http://arXiv.org/abs/chao-dyn/9705010`, 1997.

[40] A. J. Roberts. Low-dimensional modelling of dynamics via computer algebra. *Computer Phys. Comm.*, 100:215–230, 1997.

[41] A. J. Roberts. An accurate model of thin 2d fluid flows with inertia on curved surfaces. In P. A. Tyvand, editor, *Free-surface flows with viscosity*, volume 16 of *Advances in Fluid Mechanics Series*, chapter 3, pages 69–88. Comput Mech Pub, 1998.

[42] A. J. Roberts. Holistic discretisation ensures fidelity to Burgers' equation. *Applied Numerical Modelling*, 37:371–396, 2001.

[43] A. J. Roberts. Holistic projection of initial conditions onto a finite difference approximation. *Computer Physics Communications*, 142:316–321, 2001.

[44] A. J. Roberts. A holistic finite difference approach models linear dynamics consistently. *Mathematics of Computation*, 72:247–262, 2002. `http://www.ams.org/mcom/2003-72-241/S0025-5718-02-01448-5`.

[45] A. J. Roberts. Derive boundary conditions for holistic discretisations of Burgers' equation. In K. Burrage and Roger B. Sidje, editors, *Proc. of 10th Computational Techniques and Applications Conference CTAC-2001*, volume 44, pages C664–C686, March 2003. `http://anziamj.austms.org.au/V44/CTAC2001/Robe`.

[46] A. J. Roberts. Low-dimensional modelling of dynamical systems applied to some dissipative fluid mechanics. In Rowena Ball and Nail Akhmediev, editors, *Nonlinear dynamics from lasers to butterflies*, volume 1 of *Lecture Notes in Complex Systems*, chapter 7, pages 257–313. World Scientific, 2003.

[47] A. J. Roberts. A step towards holistic discretisation of stochastic partial differential equations. In Jagoda Crawford and A. J. Roberts, editors, *Proc. of 11th Computational Techniques and Applications Conference CTAC-2003*, volume 45, pages C1–C15, December 2003. [Online] `http://anziamj.austms.org.au/V45/CTAC2003/Robe` [December 14, 2003].

[48] A. J. Roberts. Resolve the multitude of microscale interactions to model stochastic partial differential equations. Technical report, `http://arxiv.org/abs/math.DS/0506533`, June 2005. Submitted to LMS Journal of Computation and Mathematics.

[49] A. J. Roberts. Computer algebra derives discretisations of the stochastically forced Burgers' partial differential equation. Technical report, [`http://www.sci.usq.edu.au/staff/robertsa/CA/caddsfbpde.pdf`], January 2006.

[50] A. J. Roberts and I. G. Kevrekidis. Higher order accuracy in the gap-tooth scheme for large-scale dynamics using microscopic simulators. In Rob May and A. J. Roberts, editors, *Proc. of 12th Computational Techniques and Applications Conference CTAC-2004*, volume 46 of *ANZIAM J.*, pages C637–C657, July 2005. `http://anziamj.austms.org.au/V46/CTAC2004/Robe` [July 20, 2005].

[51] J. C. Robinson. The asymptotic completeness of inertial manifolds. *Nonlinearity*, 9:1325–1340, 1996.

[52] S. Rosencrans. Taylor dispersion in curved channels. *SIAM J. Appl. Math.*, 57:1216–1241, 1997.

[53] G. Samaey, I. G. Kevrekidis, and D. Roose. Damping factors for the gap-tooth scheme. In S. Attinger and P. Koumoutsakos, editors, *Multiscale Modeling and Simulation*, volume 39 of *Lecture Notes in Computational Science and Engineering*, pages 93–102. Springer–Verlag, 2004.

[54] G. Samaey, I. G. Kevrekidis, and D. Roose. The gap-tooth scheme for homogenization problems. *SIAM*

Multiscale Modeling and Simulation, 4:278–306, 2005. http://epubs.siam.org/sam-bin/dbq/article/60204.

[55] G. Schöner and H. Haken. The slaving principle for Stratonovich stochastic differential equations. *Z. Phys. B—Condensed matter*, 63:493–504, 1986.

[56] M. Tutkun and L. Mydlarski. Markovian properties of passive scalar increments in grid-generated turbulence. *New J. Phys.*, 6, 2004.

[57] Eric Vanden-Eijnden. Asymptotic techniques for SDEs. In *Fast Times and Fine Scales: Proceedings of the 2005 Program in Geophysical Fluid Dynamics*. Woods Hole Oceanographic Institution, 2005. http://gfd.whoi.edu/proceedings/2005/PDFvol2005.html.

[58] S. D. Watt and A. J. Roberts. The accurate dynamic modelling of contaminant dispersion in channels. *SIAM J. Appl Math*, 55(4):1016–1038, 1995. http://epubs.siam.org/sam-bin/dbq/article/25797.

[59] M. J. Werner and P. D. Drummond. Robust algorithms for solving stochastic partial differential equations. *J. Comput. Phys*, 132:312–326, 1997.