Performance of Preliminary Test Estimator under Linex Loss Function

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ABSTRACT

This paper studies the performance of the unrestricted estimator (UE) and preliminary test estimator (PTE) of the slope parameter of simple linear regression model under linex loss function. The risk functions of both the UE and PTE are derived. The moment generating function (MGF) of the PTE is derived which turns out to be a component of the risk function. From the MGF the first two moments of the PTE are obtained and found to be identical to that obtained by using a different approach in Khan *et al.* (2002). The performance of the PTE is compared with that of the UE by using the analytical and graphical as well as the numerical methods. It is revealed that if the uncertain non-sample prior information about the value of the slope is not too far from its true value then the PTE outperforms the UE.

1 INTRODUCTION

The squared error loss (SEL) function is one of the most widely used loss functions in decision theory. The popularity of this symmetric loss function is due to its mathematical and interpretational convenience. Due to the symmetric nature it fails to differentiate between overestimation and underestimation of any parame-

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ter. The criticism against the appropriateness of the SEL is ever growing since the introduction of the asymmetric linex loss (LL) by Varian (1975).

The LL function for estimating any parameter θ by θ^*, is given by $\mathrm{L}(\delta) = b\big[\exp(a\delta) - a\delta - 1\big] \quad \forall\ a \neq 0,\ b > 0$ where δ is the estimation error. The two parameters a and b in $\mathrm{L}(\delta)$ serve to determine the shape and scale, respectively, of $\mathrm{L}(\delta)$. A positive a indicates that overestimation is more serious than underestimation and a negative a represents the reverse situation. The magnitude of a reflects the degree of asymmetry about $\delta = 0$. If $a \to 0$, then the LL reduces to the SEL. Without any loss of generality it can be assumed that $b = 0$. Further details about the properties of this loss function are available in Varian (1975), Zellner (1986), Parsian and Kirmani (2002) and Parsian and Farispour (1993).

The exclusive sample information based UE of slope parameter is uniformly minimum variance unbiased estimator. The natural expectation is that the use of additional information such as non-sample prior information with the sample information would result in a better estimator than UE. Based on both sample and non-sample prior information, Bancroft (1944) pioneered the idea of PTE and showed that with respect to SEL function it outperforms UE under certain conditions.

The main purpose of this paper is to investigate the performance of the PTE of the slope parameter of simple linear regression model under the LL function. The risk of both the UE and PTE have been derived. The MGF of the PTE is also derived in this paper. The performance of the PTE relative to that of the UE is compared. It is revealed that if the non-sample prior information about the value of the slope is not too far from its true value the PTE outperforms the UE. Otherwise, none of the estimators outperforms the other.

The layout of this paper is as follows. The model and preliminaries are presented in Section 2. The risk functions of the estimators and the first two moments of the PTE are derived in Section 3. The performances of the estimators are investigated in Section 4. Finally, some concluding remarks are presented in Section 5.

2 THE MODEL AND PRELIMINARIES

Consider a set of n random sample observations y_i for $i = 1, 2, \dots, n$ from the simple linear regression model

$$y = \beta_0 + \beta_1 x + \varepsilon \tag{1}$$

where y is the response variable, β_0 is the intercept parameter, β_1 is the slope parameter, x is the predictor and ε is the error component. Assume that the errors are independently and identically distributed as a normal variable with mean 0 and variance σ^2. In conventional notation we write ε iid $N(0,\ \sigma^2)$.

Combining sample and non-sample prior information Bancroft (1944), and later Han and Bancroft (1968), developed the preliminary test estimator (PTE) for any unknown parameter. The risk properties of this estimator, under SEL, is investigated by many authors, see for instance, Khan and Saleh (2001); Khan *et al.* (2002). Giles and Giles (1992) studied the performance of the PTE of the error variance after a pre-test of exact linear restrictions on the regression coefficients in multiple regression set-up. They compared the risk of this estimator under linex loss with that under SEL. Later Giles and Giles (1996) studied the risk of the error variance under LINEX loss after a pre-test for homoscedasticity of the variances in the two-sample heteroscedastic linear regression model.

For the linear regression model in equation (1) the exclusively sample information based unrestricted estimator (UE) of β_1 is $\tilde{\beta}_1 = S_{xx}^{-1} S_{xy}$ where $S_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})^2$ and $S_{xy} = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$. Assume that uncertain non-sample prior information about the value of the slope is available either from previous study or from practical experience of researchers or experts. Such non-sample prior information can be expressed in the form of the null hypothesis $H_0 : \beta_1 = \beta_{10}$ which may be true, but there is doubt. The estimator of β_1, under the above H_0, is known as the restricted estimator (RE), and is given by $\hat{\beta}_1^{\mathrm{RE}} = \beta_{10}$. A simple form of PTE of β_1 is

$$\hat{\beta}_1^{\mathrm{PTE}} = \tilde{\beta}_1 - (\tilde{\beta}_1 - \hat{\beta}_1^{\mathrm{RE}})\, I(F_{1,\nu} < F_{1,\nu}(\alpha)) \tag{2}$$

where $I(A)$ is an indicator function of the set A and $F_{1,\nu}(\alpha)$ is the upper α-level critical value of the F statistic with 1 and $\nu = n - 2$ degrees of freedom (d.f.)

to test the null hypothesis presented earlier. Under the alternative hypothesis, $H_a : \beta_1 \neq \beta_{10}$ the distribution of F is a non-central F with $(1, \nu)$ d.f. and non-centrality parameter Δ^2 where $\Delta = \sqrt{S_{xx}}(\beta_1 - \beta_{10})\sigma^{-1}$.

Under the SEL the PTE outperforms both the UE and RE in the neighborhood of $\Delta^2 = 0$, see for instance Khan and Saleh (2001). As Δ^2 deviates further from 0, the performance of the PTE becomes worse than those of the UE and RE. However, as Δ^2 approaches a very large value the performance of the PTE becomes the same as that of the UE. On the other hand, as Δ^2 increases from zero, the performance of the RE worsen. Therefore, with respect to SEL the PTE is regarded as an improved estimator if the value of Δ^2 is not too far from zero. However, due to the growing criticism against SEL, it is of interest to investigate the performance of the PTE under the asymmetric losses such as LL.

3 THE RISK OF UE AND PTE OF THE SLOPE

The following lemma is useful for the derivation of the risk functions of the UE and PTE.

Lemma 3.1 *If $Z \sim N(0,1)$, and Z and $S \sim \chi^2_k$ are independent then for any Borel measurable function $\phi : \Re \times (0, \infty) \to \Re$ and for any $c \in \Re$,*

$$\mathrm{E}[\exp(cZ)\phi(Z,S)] = \exp(c^2/2)\,\mathrm{E}\left[\phi(Z+c,\, S)\right] \tag{3}$$

provided $(\exp(cZ)\,\phi(Z,S))$ is integrable.

Proof. By definition

$$\begin{aligned}\mathrm{E}[\exp(cZ)\phi(Z,S)] &= \mathrm{E}[\mathrm{E}\left[\exp(cZ)\,\phi(Z,S)|S\right]] = \mathrm{E}\left[\frac{1}{\sqrt{2\pi}}\int_{\Re}\phi(z,S)\,\exp(cz - z^2/2)dz\right] \\ &= \exp(c^2/2)\,\mathrm{E}\left[\frac{1}{\sqrt{2\pi}}\int_{\Re}\phi(z,S)\,\exp(-\frac{1}{2}(z-c)^2)dz\right].\end{aligned}$$

Consider $U = Z - c$. The Jacobian of the transformation is $|J| = 1$. Therefore,

$$\mathrm{E}[\exp(cZ)\phi(Z,S)] = \exp(c^2/2)\,\mathrm{E}\left[\phi(Z+c,\, S)\right].$$

This completes the proof of the lemma.

Theorem 3.2 *The risk of the UE of β_1 under the LL function is given by*

$$\mathrm{R}\left[\tilde{\beta}_1; \beta\right] = \exp(a_1^2/2) - 1$$

where $a_1 = a\sigma/\sqrt{S_{xx}}$.

Proof. By definition, the risk function of the UE of β_1 under the LL is

$$\mathrm{R}\left[\tilde{\beta}_1; \beta_1\right] = \mathrm{E}\left[\exp(a(\tilde{\beta}_1 - \beta_1))\right] - a\,\mathrm{E}\left[\tilde{\beta}_1 - \beta_1\right] - 1\,. \qquad (4)$$

The first component of the right hand side of (4) is

$$\mathrm{E}\left[\exp(a(\tilde{\beta}_1 - \beta_1))\right] = \mathrm{E}[\exp(a_1 Z)] \qquad (5)$$

where $a_1 = a\sigma/\sqrt{S_{xx}}$ and $Z = \sqrt{S_{xx}}(\beta_1 - \beta_{10})\sigma^{-1} \sim N(0,\, 1)$.

Applying Lemma 3.1 to (5) with ϕ as identity, we get

$$\mathrm{E}\left[\exp(a(\tilde{\beta}_1 - \beta_1))\right] = \exp(a_1^2/2). \qquad (6)$$

As $\tilde{\beta}_1$ is unbiased, the second component of the right hand side of (4) is 0. Collecting the results from (6) and substituting in (4), the expression of the risk function of the UE of β_1 is obtained.

The following two lemmas are essential to derive the risk function of the PTE.

Lemma 3.3 *If X follows a non-central Student's t distribution with k d.f. and non-centrality parameter δ then*

$$f_{t\,(k,\delta)}(x) + f_{t\,(k,\delta)}(-x) = 2x\, f_{F\,(1,\,k,\,\delta^2)}(x^2) \quad \forall\ x > 0 \qquad (7)$$

where $f_{t\,(k,\delta)}(\cdot)$ is the density function of a non-central Student's t distribution with k d.f. and non-centrality parameter δ, and $f_{F\,(1,\,k,\,\delta^2)}(\cdot)$ is the density function of a non-central F distribution with $(1,\, k)$ d.f. and non-centrality parameter δ^2.

Proof. The density function of the non-central Student's t distribution with k d.f. and non-centrality parameter δ is given by

$$f_{t\,(k,\,\delta)}(x) = \frac{k^{k/2}\exp(-\delta^2/2)}{\Gamma(k/2)\sqrt{\pi}(k+x^2)^{\frac{k+1}{2}}} \sum_{i=0}^{\infty} \Gamma\left(\frac{k+1+i}{2}\right) \frac{(x\delta)^i}{i!} \left(\frac{2}{k+x^2}\right)^{i/2}. \qquad (8)$$

Consider now $f_{t\,(k,\delta)}(x) + f_{t\,(k,\delta)}(-x)$ for any arbitrary $x \geq 0$. Then (8) implies that the terms of the series with odd powers of x cancel and the terms with even powers of x are duplicated. Thus,

$$f_{t\,(k,\delta)}(x) + f_{t\,(k,\delta)}(-x) = \frac{2\,k^{k/2}\exp(-\delta^2/2)}{\Gamma(k/2)\sqrt{\pi}(k+x^2)^{\frac{k+1}{2}}} \sum_{i=0}^{\infty} \Gamma\left(\frac{k+1}{2}+i\right) \frac{x^{2i}\,\delta^{2i}}{(2i)!} \left(\frac{2}{k+x^2}\right)^i$$

$$= \frac{2k^{k/2}\exp(-\delta^2/2)}{\Gamma(k/2)\sqrt{\pi}(k+x^2)^{\frac{k+1}{2}}} \sum_{i=0}^{\infty} \Gamma\left(\frac{k+1}{2}+i\right) \frac{(x^2)^i(\delta^2)^i}{2^i\,i!\,(2i-1)!!} \frac{2^i}{(k+x^2)^i}$$

$$= 2x\frac{k^{k/2}\exp(-\delta^2/2)(x^2)^{-1/2}}{\Gamma(k/2)(k+x^2)^{\frac{k+1}{2}}} \sum_{i=0}^{\infty} \frac{\Gamma\left(\frac{k+1}{2}+i\right)}{i!\,\Gamma(\frac{1}{2}+i)} \left\{\frac{x^2\delta^2}{2(k+x^2)}\right\}^i = 2x\,f_{F(1,\,k,\,\delta^2)}(x^2).$$

This completes the proof of the lemma.

Lemma 3.4 *For any two positive integers m and n*

$$\frac{\partial f_{F(m,\,n,\,D)}(x)}{\partial D} = -\frac{1}{2}f_{F(m,\,n,\,D)}(x) + \frac{m}{2(m+2)} f_{F(m+2,\,n,\,D)}\left(\frac{mx}{m+2}\right), \quad x,\, D \in [0,\,\infty)$$

where $f_{F(l,n,D)}(\cdot)$ denotes the density function of a non-central F distribution with $(l,\,n)$ d.f. and non-centrality parameter D.

Proof. The density function of the non-central F with (m, n) d.f. and non-centrality parameter D is given by

$$f_{F(m,n,D)}(x) = \frac{\exp(-D/2)m^{m/2}n^{n/2}}{\Gamma(n/2)} \frac{x^{\frac{m}{2}-1}}{(n+mx)^{\frac{m+n}{2}}} \sum_{j=0}^{\infty} \left[\frac{mxD}{2(n+mx)}\right]^j \frac{\Gamma\left(\frac{m+n}{2}+j\right)}{\Gamma\left(\frac{m}{2}+j\right)j!}.$$

Differentiating both sides with respect to D, we get

$$\begin{aligned}
\frac{\partial f_{F(m,\,n,\,D)}(x)}{\partial D} &= -\frac{1}{2}f_{F(m,n,D)}(x) + \frac{\exp(-D/2)m^{m/2}n^{n/2}}{\Gamma(n/2)} \frac{x^{m/2-1}}{(n+mx)^{\frac{m+n}{2}}} \\
&\quad \times \sum_{j=1}^{\infty} \left[\frac{mx}{2(n+mx)}\right]^j \frac{D^{j-1}}{(j-1)!} \frac{\Gamma\left(\frac{m+n}{2}+j\right)}{\Gamma\left(\frac{m}{2}+j\right)} \\
&= -\frac{1}{2}f_{F(m,n,D)}(x) + \frac{\exp(-D/2)m^{m/2}n^{n/2}}{\Gamma(n/2)} \frac{x^{m/2-1}}{(n+mx)^{\frac{m+n}{2}}} \\
&\quad \times \sum_{i=0}^{\infty} \left[\frac{mx}{2(n+mx)}\right]^{i+1} \frac{D^i}{i!} \frac{\Gamma\left(\frac{m+2+n}{2}+i\right)}{\Gamma\left(\frac{m+2}{2}+i\right)} \\
&= -\frac{1}{2}f_{F(m,n,D)}(x) + \frac{\exp(-D/2)m^{m/2+1}n^{n/2}}{2\,\Gamma(n/2)} \frac{x^{\frac{m+2}{2}-1}}{(n+mx)^{\frac{m+2+n}{2}}} \\
&\quad \times \sum_{i=0}^{\infty} \left[\frac{mxD}{2(n+mx)}\right]^i \frac{\Gamma\left(\frac{m+2+n}{2}+i\right)}{i!\,\Gamma\left(\frac{m+2}{2}+i\right)}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2} f_{F(m,n,D)}(x) + \frac{\exp(-D/2) m^{\frac{m+2}{2}} n^{n/2}}{2\Gamma(n/2)} \frac{\left(\frac{mx}{m+2}\right)^{\frac{m+2}{2}-1} \left(\frac{m+2}{m}\right)^{m/2}}{\left\{n + (m+2)\frac{mx}{m+2}\right\}^{\frac{m+2+n}{2}}} \\
&\quad \times \sum_{i=0}^{\infty} \left[\frac{(m+2)\frac{mxD}{m+2}}{2\left\{n + (m+2)\frac{mx}{m+2}\right\}}\right]^i \frac{\Gamma\left(\frac{m+2+n}{2} + i\right)}{i!\,\Gamma\left(\frac{m+2}{2} + i\right)} \\
&= -\frac{1}{2} f_{F(m,\,n,\,D)}(x) + \frac{m \exp(-D/2)(m+2)^{\frac{m+2}{2}}\, n^{n/2} \left(\frac{mx}{m+2}\right)^{\frac{m+2}{2}-1}}{2(m+2)\Gamma(n/2)\left\{n + (m+2)\frac{mx}{m+2}\right\}^{\frac{m+2+n}{2}}} \\
&\quad \times \sum_{i=0}^{\infty} \left[\frac{(m+2)\frac{mx}{m+2}D}{2\left\{n + (m+2)\frac{mx}{m+2}\right\}}\right]^i \frac{\Gamma\left(\frac{m+2+n}{2} + i\right)}{i!\,\Gamma\left(\frac{m+2}{2} + i\right)} \\
&= -\frac{1}{2} f_{F(m,\,n,\,D)}(x) + \frac{m}{2(m+2)} f_{F(m+2,\,n,\,D)}\left(\frac{mx}{m+2}\right).
\end{aligned}
$$

This completes the proof of the lemma.

Theorem 3.5 *The risk function of the preliminary test estimator of the slope parameter β_1 under the LL function is given by*

$$
\begin{aligned}
\mathrm{R}\left[\hat{\beta}_1^{\mathrm{PTE}}; \beta_1\right] &= \exp(-a_1\Delta) G_{1,\,\nu}\left(c; \Delta^2\right) + \exp(a_1^2/2)\left[1 - G_{1,\,\nu}\left(c; (\Delta + a_1)^2\right)\right] \\
&\quad + a_1\Delta\, G_{3,\,\nu}(c/3;\, \Delta^2) - 1
\end{aligned} \tag{9}
$$

where $c = F_{1,\nu}(\alpha)$ and $G_{a,\,\nu}(q;\,\theta)$ is the cdf of non-central F distribution with (a, ν) d.f., non-centrality parameter θ and evaluated at q.

Proof. By definition, the risk function of the PTE of β_1 under the LL function is

$$
\mathrm{R}\left[\hat{\beta}_1^{\mathrm{PTE}}; \beta_1\right] = \mathrm{E}[\exp(a\Phi)] - a\,\mathrm{E}[\Phi] - 1 \tag{10}
$$

where $\Phi = \hat{\beta}_1^{\mathrm{PTE}} - \beta_1$.

The first component of the right hand side of (10) is

$$
\begin{aligned}
\mathrm{E}[\exp(a\Phi)] &= \mathrm{E}\big[\exp(a\{(\tilde{\beta}_1 - \beta_1) - (\tilde{\beta}_1 - \beta_{10})\, I(F_{1,\,\nu} < F_{1,\,\nu}(\alpha))\})\big] \\
&\quad \times \left[I(F_{1,\,\nu} < F_{1,\,\nu}(\alpha)) + I(F_{1,\,\nu} \geq F_{1,\,\nu}(\alpha))\right] \\
&= \exp(a(\beta_{10} - \beta_1)) P(F_{1,\,\nu} < c) + \mathrm{E}\big[\exp(a(\tilde{\beta}_1 - \beta_1))\, I(F_{1,\,\nu} \geq c)\big]
\end{aligned} \tag{11}
$$

where $c = F_{1,\nu}(\alpha)$.

The first component of the right hand side of (11) is

$$\exp(a(\beta_{10} - \beta_1))P(F_{1,\nu} < c) = \exp(-a_1\Delta)G_{1,\nu}(c; \Delta^2). \tag{12}$$

The second component of the right hand side of (11) can be written as

$$\mathrm{E}\big[\exp(a(\tilde{\beta}_1 - \beta_1))\, I(F_{1,\nu} \geq c)\big] = \mathrm{E}\Big[\exp(a_1 Z)\, I\Big(\frac{(Z+\Delta)^2}{\nu S_n^2/\sigma^2} \geq c\Big)\Big] \tag{13}$$

where $a_1 = \frac{a\sigma}{\sqrt{S_{xx}}}$, $Z \sim N(0,1)$ and $(\nu S_n^2/\sigma^2) \sim \chi^2_\nu$. Z and S_n are independent.

Applying Lemma 3.1 to (13) with $\phi(X,Y) = I\left(\frac{(X+\Delta)^2}{Y} \geq c\right)$ we get

$$\mathrm{E}\Big[\exp(a(\tilde{\beta}_1 - \beta_1))\, I(F_{1,\nu} \geq c)\Big] = \exp(a_1^2/2)\left[1 - G_{1,\nu}\left(c; (\Delta + a_1)^2\right)\right]. \tag{14}$$

Combining (12) and (14) the first component of the right hand side of (10) yields

$$\mathrm{E}[\exp(a\Phi)] = \exp(-a_1\Delta)G_{1,\nu}\left(c; \Delta^2\right) + \exp(a_1^2/2)\left[1 - G_{1,\nu}\left(c; (\Delta + a_1)^2\right)\right]. \tag{15}$$

From (15) the moment generating function of the PTE of β_1 is

$$\begin{aligned} m(a) = \exp(-a_1\Delta)&\left[\int_{-c}^{0} f_{t(\nu,\Delta)}(x)\,dx + \int_0^c f_{t(\nu,\Delta)}(x)\,dx\right] \\ &+ \exp(a_1^2/2)\left[1 - \int_{-c}^{0} f_{t(\nu,\Delta-a_1)}(x)\,dx - \int_0^c f_{t(\nu,\Delta-a_1)}(x)\,dx\right]. \end{aligned} \tag{16}$$

Writing $g_{t(\nu,\Delta)}(x) = f_{t(\nu,\Delta)}(x) + f_{t(\nu,\Delta)}(-x)$ for any $x > 0$ in (16) we get

$$m(a) = \exp(-a_1\Delta)\int_0^c g_{t(\nu,\Delta)}(x)\,dx + \exp(a_1^2/2)\int_c^\infty g_{t(\nu,\Delta-a_1)}(x)\,dx\,. \tag{17}$$

Applying Lemma 3.3 in (17) we get

$$m(a) = \exp(-a_1\Delta)\int_0^c f_{F(1,\nu,\Delta^2)}(y)\,dy + \exp(a_1^2/2)\int_c^\infty f_{F(1,\nu,(-\Delta-a_1)^2)}(y)\,dy\,.$$

Differentiating both sides of the above equation with respect to a, then using Lemma 3.4 and finally changing the variable $y/3$ to t in the left integral we get

$$\begin{aligned} m'(a) = \frac{\sigma}{\sqrt{S_{xx}}}\Big[&-\Delta\,\exp(-a_1\Delta)\int_0^c f_{F(1,\nu,\Delta^2)}(y)\,dy + a_1\exp(a_1^2/2) \\ &\times \int_c^\infty f_{F(1,\nu,(\Delta+a_1)^2)}(y)\,dy - (\Delta + a_1)\exp(a_1^2/2)\int_c^\infty f_{F(1,\nu,(\Delta+a_1)^2)}(y)\,dy \\ &+ (\Delta + a_1)\exp(a_1^2/2)\int_{c/3}^\infty f_{F(3,\nu,(\Delta+a_1)^2)}(y)\,dy\Big]. \end{aligned} \tag{18}$$

Putting $a = 0$ in (18) we get $\mathrm{E}(\Phi) = -(\beta_1 - \beta_{10})\, G_{3,\nu}(c/3;\, \Delta^2)$ which is the bias function or equivalently the first moments of the PTE of β_1. Therefore, the second component of the right hand side of (10) is

$$a\,\mathrm{E}[\Phi] = -a_1 \Delta\, G_{3,\nu}\left(c/3;\, \Delta^2\right). \tag{19}$$

Collecting the expressions from (15) and (19) and plugging into (10) the risk function of the PTE of β_1 under the LL function is obtained.This completes the proof of the theorem.

Differentiating both sides of (18) with respect to a, then using using Lemma 3.4 and finally putting $a = 0$ the mse function of the PTE of β_1 is obtained as

$$\mathrm{M}\left[\hat{\beta}_1^{\mathrm{PTE}}; \beta_1\right] = S_{xx}^{-1}\sigma^2[1 - G_{3,\nu}(c/3; \Delta^2) + \Delta^2\{2\, G_{3,\nu}(c/3; \Delta^2) - G_{5,\nu}(c/5; \Delta^2)\}].$$

For an equivalent expression of the mse function of the shrinkage preliminary test estimator of β_1 readers may see Khan *et al.* (2002). Similarly putting $a = 0$ in the mth order derivative of (16) the mth moment of the PTE can be obtained.

4 PERFORMANCE ANALYSIS

For any non-zero value of Δ, the risk function of the PTE of β_1 can be written as

$$\mathrm{R}\left[\hat{\beta}_1^{\mathrm{PTE}}; \beta_1\right] = \mathrm{R}\left[\tilde{\beta}_1; \beta_1\right] + g(\Delta) \tag{20}$$

where $g(\Delta) = \exp(-a_1\Delta) G_{1,\nu}\,(c; \Delta^2) + a_1\Delta\, G_{3,\nu}(c/3;\, \Delta^2) - \exp(a_1^2/2) G_{1,\nu}\big(c; (\Delta + a_1)^2\big)$. Therefore, the efficiency of the PTE relative to the UE can be written as

$$\mathrm{Eff}\left[\hat{\beta}_1^{\mathrm{PTE}}; \tilde{\beta}_1\right] = \left[\exp(a_1^2/2) - 1\right] \left[\exp(a_1^2/2) - 1 + g(\Delta)\right]^{-1}. \tag{21}$$

Under the null hypothesis, $\Delta = 0$ and hence

$$g(\Delta) = G_{1,\nu}\,(c; 0) - \exp(a_1^2/2) G_{1,\nu}\left(c; a_1^2\right) < 0 \quad \forall\; a \neq 0\,. \tag{22}$$

Therefore, at $\Delta = 0$ the PTE is more efficient than the UE.

For any positive a, if Δ is positive $a_1\Delta\, G_{3,\nu}(c/3;\, \Delta^2)$ is also positive. Therefore, for positive a as Δ grows larger from 0 efficiency of PTE decreases and crosses the 1-line at $\Delta = (\exp(a_1^2/2)G_{1,\nu}\,(c;(\Delta+a_1)^2) - \exp(-a_1\Delta)G_{1,\nu}\,(c;\Delta^2))/(a_1\, G_{3,\nu}(c/3;\, \Delta^2))$ regardless of the value of a.

For any negative a, if Δ is positive $a_1\Delta\, G_{3,\nu}(c/3;\, \Delta^2)$ is negative. Therefore, for negative a, as Δ grows larger from 0, the efficiency of PTE grows larger, reaching its maximum at some Δ depending on the magnitude of a, and then starts decreasing and crosses the 1-line for the value of Δ given above regardless of the value of a. As

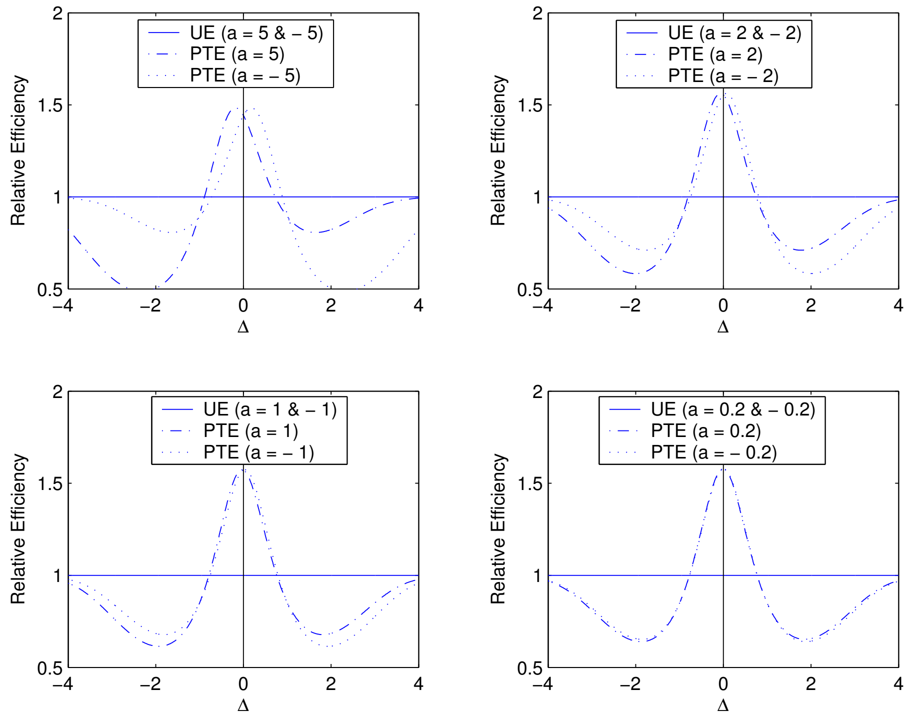

Figure 1: Efficiency of PTE relative to UE for $\alpha = 0.2$, $n = 25$ and selected a.

$\Delta \to \infty$, $g(\Delta)$ tends to 0, and hence, Eff $\left[\hat{\beta}_1^{\text{PTE}}; \tilde{\beta}_1\right] \to 1$. Therefore, starting from a certain large Δ, the efficiency of PTE is no different from that of UE.

For very small values of a, the growth pattern of the efficiency of the PTE for both positive and negative values of Δ are very similar. Because for very small values of a, the LL function reduces to the SEL function.

From the foregoing analyses and Figure 1 it is clear that the efficiency of the PTE relative to the UE depend on the three factors, the values of α, Δ and a. The

Table 1: Maximum and minimum efficiencies of PTE relative to UE for $a = 3$

α		Sample size, n						
		10	15	20	25	30	35	40
0.05	Eff*	4.2674	3.9892	3.8713	3.80640	3.7653	3.7369	3.7162
	Eff$_o$	0.2661	0.3076	0.3279	0.3279	0.3401	0.3401	0.3545
	Δ_o	−2.6300	−2.4700	−2.3850	−2.3950	−2.3550	−0.3250	−2.3004
0.10	Eff*	2.5615	2.4468	2.3978	2.3706	2.3534	2.3415	2.3328
	Eff$_o$	0.3807	0.4202	0.4393	0.4507	0.4585	0.4641	0.4684
	Δ_o	−2.3550	−2.2255	−2.1515	−2.1448	−2.1050	−2.1105	−2.0850
0.15	Eff*	1.9556	1.8907	1.8629	1.8474	1.8376	1.8308	1.8258
	Eff$_o$	0.4748	0.5112	0.5285	0.5390	0.5462	0.5511	0.5550
	Δ_o	−2.1750	−2.0850	−2.0390	−2.0100	−2.0055	−2.0025	−1.9750
0.20	Eff*	1.6429	1.6014	1.5835	1.5736	1.5673	1.5629	1.5597
	Eff$_o$	0.5573	0.5900	0.6055	0.6148	0.6211	0.6256	0.6291
	Δ_o	−2.0610	−1.9800	−1.9500	−1.9302	−1.9100	−1.9000	−1.8950
0.25	Eff*	1.4530	1.4247	1.4125	1.4057	1.4014	1.3984	1.3962
	Eff$_o$	0.6310	0.6597	0.6733	0.6814	0.6868	0.6908	0.6938
	Δ_o	−1.9850	−1.9250	−1.8950	−1.8755	−1.8609	−1.8550	−1.8500
0.30	Eff*	1.3268	1.3068	1.2982	1.2934	1.2904	1.2883	1.2867
	Eff$_o$	0.6969	0.7215	0.7331	0.7400	0.7446	0.7480	0.7506
	Δ_o	−1.925	−1.8650	−1.8459	−1.8255	−1.8100	−1.8080	−1.7968

value of a is determined by the experimenter according to the potential impact of the positive and negative errors of estimation, and the value of Δ is usually unknown to the experimenter. Regardless of the values of a and Δ, the efficiency of the PTE is a function of α. The question is which value of α should be used for the preliminary test?

Let us consider the efficiency function of the PTE of β_1 relative to the UE as a function of α and Δ. Therefore,

$$\text{Eff}\left[\hat{\beta}_1^{\text{PTE}}; \alpha, \Delta\right] = \left[\exp(a_1^2/2) - 1\right]\left[\exp(a_1^2/2) - 1 + g(\Delta)\right]^{-1}. \tag{23}$$

From the analyses of the relative efficiency function of the PTE it is evident that the PTE does not have uniform domination over the UE for all values of Δ. Also, the value of Δ is usually unknown to the experimenter. Thus, we pre-assign a value of the relative efficiency, say Eff_o, that we are willing to accept. Consider the set $A_\alpha = \left(\alpha \mid \mathrm{Eff}\left[\hat{\beta}_1^{\mathrm{PTE}}; \alpha, \Delta\right] \geq \mathrm{Eff}_o\right)$ for all Δ. An estimator $\hat{\beta}_1^{\mathrm{PTE}}$ is chosen which maximizes $\mathrm{Eff}\left[\hat{\beta}_1^{\mathrm{PTE}}; \alpha, \Delta\right]$ $\forall\ \alpha \in A_\alpha$ and Δ.Thus we solve

$$\max_\alpha \min_\Delta \mathrm{Eff}\left[\hat{\beta}_1^{\mathrm{PTE}}; \alpha, \Delta\right] = \mathrm{Eff}_o \tag{24}$$

for α. The solution provides the maximum and minimum guaranteed efficiencies of the PTE of β_1 relative to the UE, for any selected values of n and Δ. Tables 1 presents the maximum guaranteed efficiency (Eff^*) and minimum guaranteed efficiency (Eff_0) of the PTE of β_1 relative to the UE, and the value of Δ (Δ_o) at which Eff_0 occurs, for selected values of a, n and α. For example, if $a = 3$ and $n = 20$, and the experimenter wishes to achieve the minimum guaranteed efficiency 0.6055 of the PTE of β_1, the recommended value of α is 0.20. This minimum guaranteed efficiency attains at $\Delta_o = -1.95$. For $a = -3$ the same minimum guaranteed efficiency occurs at $\Delta_o = 1.95$. In general, if a is negative the value of Δ_o is positive and vice-versa.

5 CONCLUDING REMARKS

This paper has introduced the computation of any order derivative of the non-central F distribution with respect to the non-centrality parameter and the derivation of the MGF of the PTE. Moment of the PTE of any order can be obtained from this MGF. For illustration, the bias and mse functions are derived. It is revealed that if the non-sample prior information regarding the value of the parameter is not too far from its true value the PTE outperforms the UE. This result reaffirms the superiority of the PTE under the SEL function. Similar to the shape of the LL function the shape of the risk function of the PTE is also asymmetric. As the value of the shape parameter of the loss function grows smaller the shape of the risk function of the PTE approaches symmetry.

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References

Bancroft, T.A. (1944) On biases in estimation due to the use of the preliminary test of significance. *Ann. Math. Statist.*, **15**, 190-204.

Giles, J.A. and Giles, D.E.A. (1996) Risk of a homoscedasticity pre-test estimator of the regression scale under LINEX loss. *J. Stat. Plan. Inf.*, **50**, 21-35.

Giles, J.A. and Giles, D.E.A. (1992) Preliminary-test estimation of the regression scale parameter when the loss function is asymmetric. *Comm. Statist., Theo. and Meth.*, **22**(6), 1709-1733.

Han, C.P. and Bancroft, T.A. (1968) On pooling means when variance is unknown. *J. Am. Stat. Asso*, **63**, 133-1342.

Khan, S., Hoque, Z. and Saleh, A.K.Md.E. (2002) Estimation of the slope parameter for linear regression model with uncertain prior information. *J. Statist. Res.*, **36**(1), 55-73.

Khan, S. and Saleh, A.K.Md.E. (2001) On the comparison of the pre-test and shrinkage estimators for the univariate normal mean. *Statist. Pap.*, **42**(4), 451-473.

Parsian, A. and Kirmani, S.N.U.A. (2002) Estimation under LINEX Loss Function. In *Handbook of Applied Econometrics and Statistical Inference*, eds. A. Ullah, A.T.K. Wan, and A. Chaturvedi. New York: Marcel Dekker.

Parsian, A. and Farispour, N.S. (1993) On the admissibility and inadmissibility of estimators of scale parameters using an asymmetric loss function, *Comm. Statist., Theo. and Meth.*, **22**(10), 2877-2901.

Varian, H.R. (1975) A Bayesian approach to real estate assessment. In *Studies in Bayesian Econometrics and Statistics in honor of L.J. Savage, eds., Feinberg and A. Zellner*, North Holland, Amsterdam, 195-208.

Zellner, A. (1986) Bayesian estimation and prediction using asymmetric loss functions. *J. A. Statist. Ass.*, **81**(394), 446-451.