Dynamic behaviour of laterally loaded pultruted composite beams

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Abstract. This paper discusses a detailed study carried out on vibration characteristics of transversely loaded pultruted composite beams. A considerable discrepancy was noticed in the measured vibration of the transversely loaded beam and the calculated natural frequencies of the beam using Euler-Bernoulli beam theory. It has been observed that the natural frequencies of the beam were increased after lateral loading. A simplified mathematical model was derived for the equation of motion in order to determine the vibration characteristics of the loaded beam. The discontinuities introduced at the loading points were modelled as a combination of translational and torsional springs. The predictions of the model have demonstrated that it has sufficiently captured the dynamic behaviour of a laterally loaded beam. A comparison of model prediction and the experimental results is presented.

1. INTRODUCTION

In the past few decades, advanced fibre composite materials have revolutionized the fields of aerospace, marine, energy and civil infrastructure industries. Due to high stiffness, and strength-to-weight ratio, considerable fatigue life and resistance to chemical corrosion are the major factors of the success in composite materials in many expensive and critical infrastructure constructions.

The expansion of composites in the industry is facing new challenges due to imposed standards in the field. Advanced fibre composite structures which are being used in defence, aerospace and civil infrastructures suffer harsh static and dynamic loading which will degrade material properties, and cause disintegration of the structure and catastrophic failures. As such, there is a growing demand for a reliable and an accurate structural health monitoring (SHM) system to maintain the structural integrity and extended life-span of these expensive and critical advanced composite structures. Structural health monitoring systems principally use static and dynamic responses of structures for the purpose of detecting damage and estimating residual life. Vibration techniques have been used in the aerospace industry for a few decades to detect damage in composite structures [1, 2] due to the simplicity of implementing dynamic response based damage detection methods. Consequently, it is important to establish an in-depth knowledge in dynamic response of structural components frequently used in advance composite constructions before implementing a vibration based SHM system.

Pultruted profiles are becoming more popular in advanced composite structural construction, due to the ease of manufacturing, good external finish, good dimensional tolerance and the excellent mechanical properties. In recent years pultruted profiles are largely employed in aerospace structural applications such as deck beams, vertical stabilizers etc. The current research conducted on advanced

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composite structures at the Centre of Excellence in Engineered Fibre Composites (CEEFC) of the University of Southern Queensland has developed hybrid composite bridges chiefly using pultruted beam sections to replace wooden bridges in rural Australia. As such, an investigation of the dynamic behaviour of pultruted sections will be absolutely useful for the development of SHM systems for advanced composite structures.

2. OBJECTIVE

Advanced composite beams experience various types of loading at different orientations, according to their arrangement in the structure. The most common loading type for beams is lateral loading such as live loads on floor beams and bridge girders.

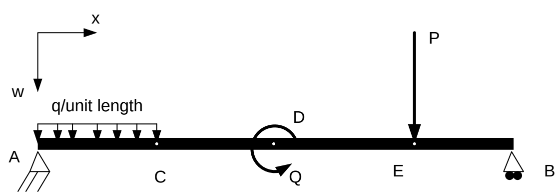


Figure 1. Typical laterally loaded beam

Consider the laterally loaded beam shown in Figure 1. From the first principles, each loading point corresponds to a discontinuity point in the elastic curve. The radius of curvature (*R*) changes at each discontinuity point:

$$\frac{1}{R} = \frac{\partial^2 w(x,t)}{\partial x^2} \tag{1}$$

since the lateral deflection $w(x)$ changes with the bending moment $M(x)$ as:

$$EI\,\frac{\partial^2 w(x,t)}{\partial x^2} = M\,(x,t) \tag{2}$$

As a consequence, each part of the beam bounded by loading points and the supports would behave differently under dynamic excitation while satisfying the common boundary conditions such as displacement, slope, shear force and the bending moments at C, D and E.

The natural frequencies of beams with various boundary conditions have been considered by many researchers [3, 4, 5]. These analyses were based on Euler-Bernoulli and Timoshenko beam theory and exact and approximate solutions have been obtained for beams with attached masses, spring masses etc. However, no attention has been given to the discontinuities caused by point loads on beams.

This paper describes the research work undertaken to investigate the effects of lateral loads on the dynamic response of pultruted beams. The experimental procedures and the analytical method will be discussed in following sections.

3. ANALYSIS

3.1 Vibration model

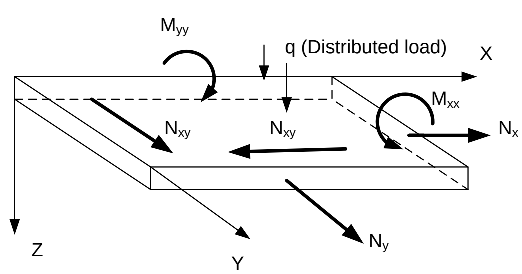


Figure 2. A typical plate element

Starting from the classical laminated plate theory (CLTP) for composite plate (Figure 2), a governing equation of motion can be defined as [6]:

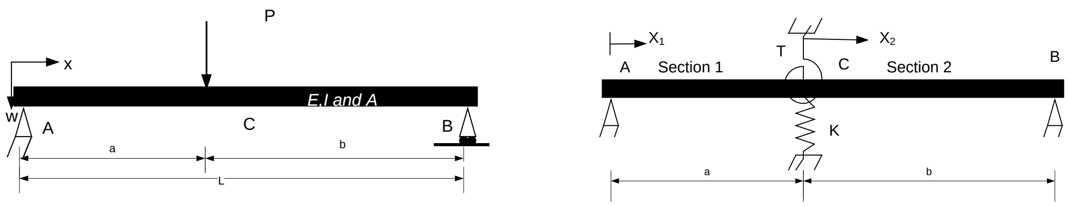
$$\frac{\partial^2 M_{xx}}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 Myy}{\partial y^2}+N(w)+q=\rho\frac{\partial^2 w}{\partial t^2}-\bar{I}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+\bar{Z}\frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)$$

$$where \quad N(w)=\frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}+N_{xy}\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}+N_{yy}\frac{\partial w}{\partial y}\right) and \qquad (3)$$

$$\bar{Z}=\int_{-h/2}^{h/2}\rho\, z\, dz \quad and\ h-thickness\ of\ the\ plate,\ \rho-density\ of\ the\ layer\ material$$

When the width of the plate, y, is small compared to its length along the x–axis, and the loading and transverse displacements are functions of x only, then the plate can be treated as a beam. As a consequence, one dimensional (1D) analysis would be sufficient to obtain dynamic characteristics. For 1D analysis, neglecting $\boldsymbol{y}$ terms and the rotary inertia term, equation (3) can be re-written as:

$$EI\frac{\partial^4 w}{\partial x^4}(x,t)\ +\rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t)=0 \qquad (4)$$



(a) **(b)**

Figure 3. (a) Laterally loaded beam **(b)** Equivalent beam with a translational and a torsional spring

The equation of motion for a pultruted composite beam (equation (4)) is similar to Euler-Bernoulli equation of motion for an isotropic beam.

Assuming solution for equation (4) as $w(x,t)=W(x)e^{i\omega t}$ where ω is natural frequency of the beam. Substituting into equation (4):

$$EI\frac{d^4 W}{dx^4}-\omega^2\rho A=0 \qquad (5)$$

Assuming general solution for equation (5) as:

$$\boldsymbol{W(x) = c_1 Sin(\lambda x)+ c_2 Sinh(\lambda x)+ c_3 Cos(\lambda x)+ c_4 Cosh(\lambda x)} \qquad (6)$$

where c_1, c_2, c_3, c_4 are constants

Assuming the solution for equation (5) as $\omega=\lambda^2\sqrt{\frac{EI}{\rho A}}$ and using simply supported boundary conditions, i.e. **@ x=0, $w(0)$=0 and , @ x=L, $w(L)$=0, λ** can be calculated as:

λ=*nπ/L* *for n=1…….n.*

The above equation gives the natural frequencies of the beam AB which has no discontinuities. However, the beam AB has a discontinuity due to the load acting on point C which has not been considered in the formulation. Therefore, the above natural frequency solution is no longer valid for the loaded beam AB. To incorporate the loading point discontinuity, the following procedures will be adopted in the model development.

The discontinuity at point C can be considered as a boundary and the load can be replaced as combined torsional and translational springs as shown in Figure 3(b) [3]. Consequently, the beam AB becomes two interconnected beams, AC and BC. Equation (5) can then be applied to the beam segments AC and BC. The boundary conditions for the continuity at point C must be satisfied by both beam segments.

For the continuity of the beam at C, the following boundary conditions must be satisfied:

$$\begin{aligned}
&\textit{Displacement} - \quad [w_1(x_1)\big|_{x_1=a} = w_2(x_2)\big|_{x_2=0} \\
&\textit{Slope} - \quad \left.\frac{dw_1(x_1)}{dx_{i-1}}\right|_{x_1=a} = \left.\frac{dw_2(x_2)}{dx_2}\right|_{x_2=0} \\
&\textit{Shear force} - \quad EI\left(\left.\frac{d^3w_1(x_1)}{dx_1^3}\right|_{x_1=a}\right) = EI\left(\left.\frac{d^3w_2(x_2)}{dx_2^3}\right|_{x_2=0}\right) + K\,w(x_2)\big|_{x_2=0} \\
&\textit{Bending moment} - \quad EI\left(\left.\frac{d^2w_1(x_1)}{dx_1^2}\right|_{x_1=a}\right) = EI\left(\left.\frac{d^2w_2(x_2)}{dx_2^2}\right|_{x_2=0}\right) + T\frac{dw(x_2)\big|_{x_2=0}}{dx_2}
\end{aligned} \tag{7}$$

where K and T are the stiffness of translational and rotational springs respectively.

As such, for two segments of the beam AB there are 12 homogeneous equations that can be developed. Since, there are two rigid supports that have no adjacent beam segments at A and B, the total number of equations can be reduced to 6. Bapat and Bapat [3] has shown that a single value of $\boldsymbol{\lambda}$ per mode satisfies the solution of the equation of motion of both segments of the beam. Thus neglecting the torsional effect (T) for small deflections, the continuity boundary conditions give the frequency equation as:

$$\begin{vmatrix}
Sin(\lambda a) & Sinh(\lambda a) & -1 & 0 & -1 & 0 \\
Cos(\lambda a) & Cosh(\lambda a) & 0 & -1 & 0 & -1 \\
-Sin(\lambda a) & Sinh(\lambda a) & 1 & 0 & -1 & 0 \\
-\lambda^3 Cos(\lambda a) & \lambda^3 Cosh(\lambda a) & -\dfrac{K}{EI} & 0 & -\dfrac{K}{EI} & 0 \\
0 & 0 & Cos(\lambda b) & Sin(\lambda b) & 0 & 0 \\
0 & 0 & 0 & 0 & Cosh(\lambda b) & Sinh(\lambda b)
\end{vmatrix} = 0 \tag{8}$$

Solving the determinant above (Equation (8)), the natural frequencies of the beam segments can be calculated.

3.2 Finite Element Analysis (FEA) of the loaded beam

A FEA mesh was created for a pultruted beam of 50mm x 50 mm x 5mm and 1300 mm long using STRAND7 commercial software. The model comprised with 12000 QUAD4 plate elements. The model was solved using dynamic solver and post processed to obtain natural frequencies and the mode shapes.

Natural frequencies of the loaded beam were calculated first. Thereafter, the loading point of the beam was replaced with constraint in y direction (loading direction) and then the natural frequencies of the beam were re-calculated.

4. EXPERIMENTATION

A 50mm x 50 mm x 5mm and 1300 mm long pultruted section was simply supported on a 3 point bending rig of span 900 mm. The supported beam was mounted on a 100kN MTS universal testing machine. The beam was loaded from 0 to 5 kN in a few steps. A three axis MEM ADXL330 accelerometer was attached to the top surface of the beam at a location of 100mm from the mid point of the beam. A PCB086C04 impulse hammer was used to excite the system and LMS VB8 front end was used for data acquisition at a maximum rate of 25kHz. The data was post processed using LMS Testxpress© software.

5. RESULTS AND DISCUSSION

Table 1. Calculated λ for various values of normalized stiffness ***k***

Mode	***k =Ka³/EA***					
	10	10^2	10^3	10^4	10^8	10^{12}
1	1.483290	1.741530	3.141600	3.141600	3.141600	3.1416
2	2.243390	3.044190	3.640510	3.707660	3.714070	3.71477
3	3.141600	3.141600	4.841310	4.916880	4.923660	4.92439
4	4.206000	4.368150	6.283190	6.283190	6.283190	6.28319

Table 2. Experimental natural frequencies in Hz at various loads and FEA results for the beam constrained at middle (loading point).

Mode	FEA (Mid Constrained)	LOAD (kN)				
		0	1	2	3	5
1	395	124	416	420	424	416
2	437	448	650	446	458	710
3	648	842	712	694	806	748
4	720	-	756	796	-	-

Table 3. Calculated frequencies in Hz for various normalized stiffness ***k*** (using λ from Table 1)

Mode	Normalized stiffness (*k*)					
	10	10^2	10^3	10^4	10^8	10^{12}
1	130	180	584	584	584	584
2	298	549	785	814	817	817
3	584	584	1388	1431	1436	1436
4	1047	1130	2337	2337	2337	2337

Table 4. Calculated, FEA and measured frequencies in Hz for beam with no load

Mode	Experimental	FEA	Euler-Bernoulli
1	124	137	146
2	448	396	584
3	842	649	1315
4	1735	990	2337

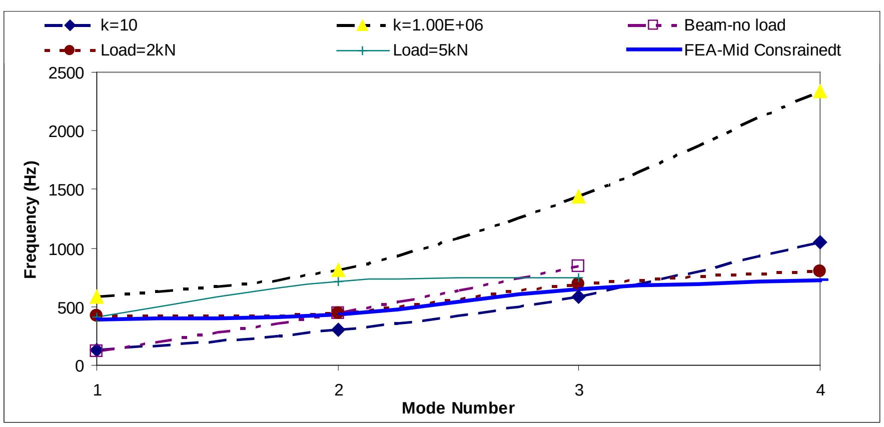


Figure 4. Graph of natural frequencies obtained analytically and experimentally

Tables 1 to 4 and Figure 4 show the analytical and experimental natural frequencies of a laterally loaded simply supported beam.

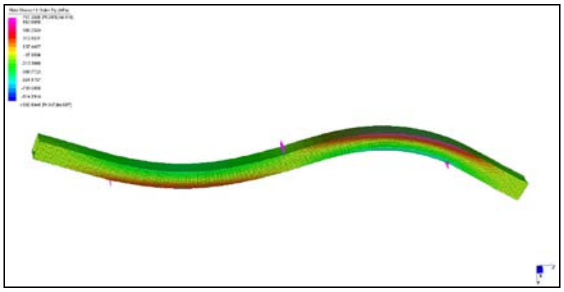


Figure 5. FEA results Mode 1 – 395 Hz

Figure 5 illustrates the first mode of the beam with vertical constrained at the mid span. The experimental results clearly show that the first natural frequency of the loaded beam increased by approximately four fold irrespective of the load intensity (Table 2). FEA results of the beam constrained at the loading points (mid. span) shows a similar trend. This frequency is approximately equal to the first natural frequency of a beam which has half of the length of the beam considered here. The calculated frequencies (Table 3) show that with the increase of stiffness (k) of the assumed translational spring will increase the natural frequencies of the beam. However, the calculated natural frequencies do not change significantly when $k > 10^4$. The calculated natural frequencies for large k ($10 < k < 10^4$) values show a similar trend as experimental/FEA results, but the calculated frequencies are higher than the experimental results.

The frequencies listed in Table 4 show some discrepancies between calculated and experimental natural frequencies for the unloaded beam. The natural frequencies calculated from Euler-Bernoulli beam theory show a considerable deviation from the experimentally obtained natural frequencies of mode 3 and above. However, the FEA results show some correlation with experimental results.

6. CONCLUSIONS

Dynamic response of a beam loaded at its mid-span was investigated experimentally and analytically. The natural frequencies of the beam have increased considerably due to the lateral loading. A mathematical model was developed in order to study the dynamic response of the loaded beam. The analysis has shown that the assumption of discontinuity at the loading point as a combination of translational and rotational springs is reasonable and the model predicted alike results to experimental results. The analytical results show that the developed mathematical model has reasonably captured the dynamic behaviour of a laterally loaded beam. However, further work needs to be done to fine tune the model to address the discrepancies between experimental results and the model predictions. An investigation into the feasibility and the limitations of Euler-Bernoulli beam theory for pultruted beam is also warranted.

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