Two dimensional particle solution of the extended Hamilton–Jacobi equation

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Abstract

In classical mechanics the Hamilton–Jacobi equation for a free particle has the property of reducing a perturbation of spatially uniform solution into a point. In the late 1970s Sivashinsky proposed an extension of the equation so that it takes the form of the Kuramoto–Sivashinsky equation under which a smooth soliton is formed instead of the point. The soliton was proposed as a model for spatially extended elementary particle. However, this solution is unstable. Developing Sivashinsky's idea further, we propose a different extension which ensures stability. We performed two dimensional computational experiments demonstrating the soliton formation and stability.

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1 Introduction

Describing elementary particles as localised structures of continuous fields is a challenging problem since the work of de-Broglie [1]. Sivashinsky [2, 3] noticed mathematical similarity between the relativistic Hamilton–Jacobi (HJ) equation for a free particle in classical mechanics,

$$(1/c^2)S_t^2 = (\nabla S)^2 + m^2c^2\,, \tag{1}$$

and the equation describing hydrodynamic instability of laminar flames [4],

$$(1/\nu^2)H_t^2 = (\nabla H)^2 + 1\,. \tag{2}$$

In the HJ equation, (1), $S = S(x, y, z, t)$ is the action function, c is the speed of light and m is the mass of the particle. The gradient of S has the dimension of momentum and time derivative of S has the dimension of energy. In the flame front equation, (2), $H = H(x, y, t)$ is the distance travelled by the front along a tube as a function of the transversal coordinates x and y and time, and ν is the constant having the dimension of velocity.

Clearly equation (2) admits the solution describing a flat front moving with the speed ν:

$$H_* = \nu t\,. \tag{3}$$

Consider weak perturbations h of (3), defined by $H(x, y, t) = \nu t - h(x, y, t)$. Characterized by small gradients they evolve according to

$$1 - (1/\nu)h_t = \sqrt{1 + (\nabla h)^2} \approx 1 + (1/2)(\nabla h)^2 ;$$

that is, approximately as

$$h_t = -\frac{\nu}{2}(\nabla h)^2 . \tag{4}$$

Similarly, small perturbations s of the flat solution of (1),

$$S_* = -mc^2\, t\,, \tag{5}$$

obey the equation

$$s_t = -\frac{1}{2m}(\nabla s)^2 \tag{6}$$

referred to as the non-relativistic HJ equation. The similarity of the equations underpins similarity of solutions. Under (4) and (6) the perturbation shrinks into a point that is an object with zero spatial extension.

Sivashinsky extended the flame equation (4) by including linear terms:

$$h_t = -\frac{1}{2}(\nabla h)^2 - \nabla^2 h - \nabla^4 h \tag{7}$$

well known as the Kuramoto–Sivashinsky equation [4]. The balance between the source, $-\nabla^2 h$, and dissipation, $-\nabla^4 h$, bridged by the nonlinearity, $-(1/2)(\nabla h)^2$, results in a smooth dissipative structure instead of a point.

By analogy, Sivashinsky extended the HJ equation (1) [2, 3] making sure that Lorentz invariance is preserved so that the extended version is suitable for the relativistic case:

$$\frac{1}{c^2}S_t^2 = (\nabla S)^2 + m^2c^2 + \alpha\hbar\,\Box S + \beta\frac{\hbar^3}{m^2c^2}\,\Box^2 S\,, \tag{8}$$

where α and β are positive non-dimensional constants, $\Box = \nabla^2 - (1/c^2)\partial_t^2$. Equation (8) admits a smooth soliton-type solution, which could be used to represent the particle 'smeared' in space over some characteristic distance Δr. The ratio of the amplitude of the soliton, Δs, to the width, Δr, would then define the characteristic momentum of 'self-vibrations' of the particle. The amplitude, Δs, must satisfy the uncertainty principle, $\Delta s \sim \Delta p\, \Delta r \sim \hbar$, where $\hbar$ is Plank's constant.

However, such a soliton is unstable with respect to small perturbations. Because of the *linear* nature of the source, long wave perturbations of the platform on which the soliton rests are amplified and the whole structure is destroyed. In an attempt to overcome instability we modified the equation (8) further [5, 6]:

$$\frac{1}{c^2}S_t^2 = (\nabla S)^2 + m^2c^2 - \alpha\hbar\Box S + \beta\frac{\hbar^3}{m^2c^2}\Box^2 S - \lambda\frac{\hbar^2}{m^2c^2}(\Box S)^2 - \mu\frac{\hbar^3}{m^4c^4}(\Box S)^3 . \tag{9}$$

Here λ and μ are positive non-dimensional constants. See that the sign in front of $\alpha\hbar\Box S$ is opposite to the respective sign in (8); this converts the term from source into dissipation. The new quadratic nonlinear term is introduced to restore the source. It is counterbalanced by the cubic nonlinearity.

For certain range of the coefficients and under the periodic boundary conditions, we numerically obtained solutions in the form of train of solitons [5]. By stretching the period an isolated stationary soliton is obtained.

2 Has stability been achieved?

Sivashinsky brought to my attention [7] that this soliton is still unstable to certain perturbations. Indeed, the linearised equation (9) has the form

$$2s_t - \alpha s_{xx} + \alpha s_{tt} + \beta s_{xxxx} + \beta s_{tttt} - 2\beta s_{ttxx} = 0 .$$

Substituting $s \sim \exp(ikx + \omega t)$ leads to the dispersion relation

$$2\omega + \alpha\omega^2 + \beta\omega^4 + 2\beta\omega^2 k^2 = -\alpha k^2 - \beta k^4 \,. \tag{10}$$

Consider flat perturbations, $k = 0$. Then (10) becomes

$$2\omega + \alpha\omega^2 + \beta\omega^4 = 0 \,. \tag{11}$$

Obviously (11) has a neutral root, $\omega = 0$, but also there is a pair of complex-conjugate roots with real parts proportional to

$$-3\beta\left(-\frac{1}{\beta} + \frac{1}{3\beta}\sqrt{\frac{\alpha^3}{3\beta} + 9}\right)^{2/3} + \alpha \,.$$

It can be shown that this expression is always positive, which implies instability. The arguments [5] in favour of stability of the soliton generated by (9) are only applicable to long wave perturbations with $\omega \approx -(\alpha/2)k^2$ which is a root of (10). This root transforms into the neutral root $\omega = 0$ of (11) if the terms of order k^2 and k^4 in (10) are neglected and (11) is used as an approximation of (10).

3 Stable model

To overcome instability we abandon the Lorentz invariance, thus restricting attention to slow motions:

$$\begin{aligned} S_t &= -\frac{1}{2m}(\nabla S)^2 - mc^2 + \alpha\hbar\nabla^2 S - \frac{\beta\hbar^3}{m^3c^2}\nabla^4 S \\ &\quad + \frac{\lambda\hbar^2}{m^3c^2}(\nabla^2 S)^2 + \frac{\mu\hbar^3}{m^5c^4}(\nabla^2 S)^3 \,. \end{aligned} \tag{12}$$

While the Lorentz invariance is sacrificed in (12), Galilean invariance is preserved due to the Galilean invariance of the classical part of (12) and chosen form of the added terms. Indeed, suppose that S obeys the classical law (6), then, in the moving coordinates $t_* = t$, $x_* = x - Vt$, $y_* = y$ (Galilean transformation), the action $S_* = S - mVx_* - \frac{1}{2}mV^2t_*$ obeys the same classical law. The additional momentum, mV, and energy, $\frac{1}{2}mV^2$, emerge because of the moving coordinates. Likewise, when we postulate the extended HJ equation (12) for S, then the action S_* obeys the same extended equation in the moving system. Since the shift of S_* relative to S is linear in x_* and t_*, the terms with the operators ∇^2 and ∇^4 do not destroy the invariance.

4 A simpler model

The linearised version of (12) is

$$2S_t = \alpha S_{xx} - \beta S_{xxxx} . \tag{13}$$

Observe that both the second and fourth derivatives represent dissipation (this ensures stability of the plateau on which the soliton stands). However, there is no apparent reason why the model needs both terms. Yet, in our previous 1D numerical experiments, we never observed stationary solitons when only one dissipative term is used, that is, when either $\alpha = 0$ or $\beta = 0$.

In this article we ask ourselves whether in the 2D case, which is arguably more realistic than 1D case, one of the two dissipations can be spared? this would make the model simpler. Numerical results presented in the next section give a positive answer to this question. Thus, we remove the second spatial derivative from the model (12). Performing the shift $S = \hat{S} - mc^2t$ and transforming to the non-dimensional quantities,

$$S_1 = \hat{S}/\hbar , \quad t_1 = t\, mc^2/\hbar , \quad x_1 = x\, mc/\hbar , \quad y_1 = y\, mc/\hbar ,$$

we arrive at the non-dimensional equation (hereafter the subscript 1 is omitted),

$$S_t = -\frac{1}{2}(\nabla S)^2 - \beta\nabla^4 S + \lambda(\nabla^2 S)^2 + \mu(\nabla^2 S)^3 . \tag{14}$$

5 Numerical experiments

We solved equation (14) numerically in a square domain (x, y). The equation was discretized in space using finite differences on a uniform grid with 60×60 grid points and step size $10/3$. The system of ODEs was resolved in time using the MATLAB solver `dae2.m` [8] under the boundary conditions of zero normal derivatives on the edges of the domain. As the initial condition we seeded a small peak in the centre of the domain. For convenience of computations the time was rescaled, $t \to t\tau$, involving the non-dimensional coefficient τ. This allowed us to transform the right-hand side of (14) to a form where the coefficients at all terms were available for variation:

$$S_t = -(\tau/2)(\nabla S)^2 - \beta\tau\nabla^4 S + \lambda\tau(\nabla^2 S)^2 + \mu\tau(\nabla^2 S)^3 .$$

A sufficiently small peak decays under the linear dissipation. However, when large enough, the peak grows pumped up by the source $\sim (\nabla^2 S)^2$. Gradually the growth halts under the stabilizing effect of the cubic term $\sim (\nabla^2 S)^3$ (see Figure 1) and a stationary soliton is formed. This scenario realises for some range of the equation's coefficients. We do not aim to determine the range exactly at this stage; more important is that this range exists.

Compare contributions of different terms into the balance. Figure 2 shows that the peak is largely maintained by the balance between the nonlinear source and nonlinear stabilizer. A small gap between them is compensated by the quadratic term, $\sim -(\nabla S)^2$. Inside the peak area, the linear dissipation is negligible compared to the nonlinear terms. In contrast, outside the peak the dissipation dominates. This property ensures stability of the platform on which the peak rests.

Outside the range of the coefficients leading to the stationary soliton, we observed various nonstationary regimes. For example, for fixed $\beta\tau$, $\lambda\tau$ and $\mu\tau$ but different τ we found that, for larger τ, the peak moves downward.

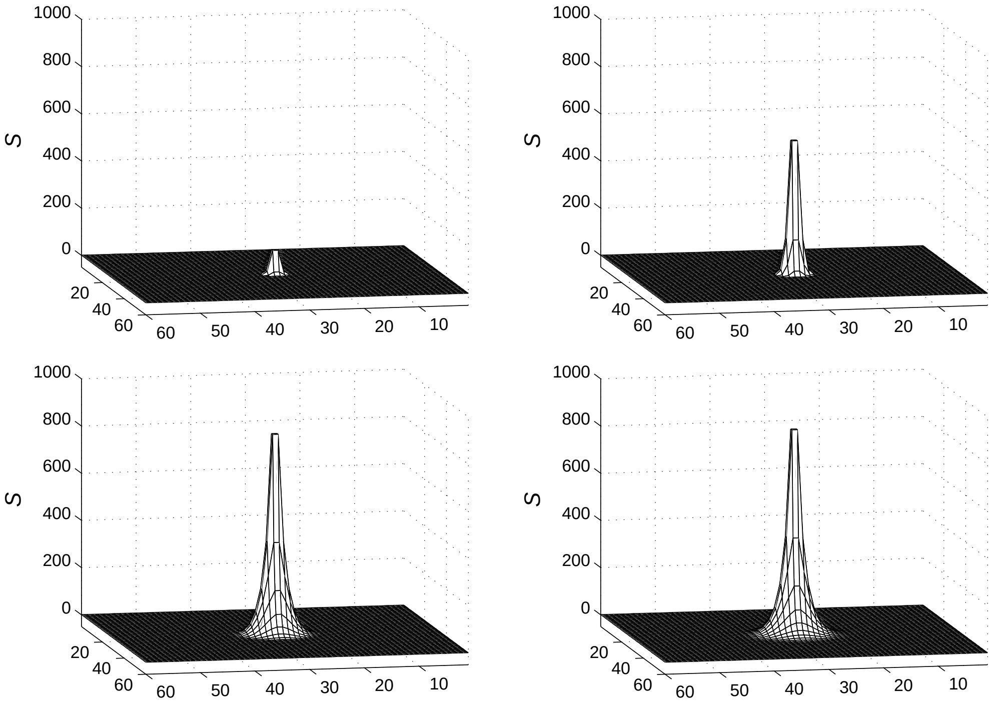

Figure 1: The formation of the stationary soliton. $t = 0.425$, 0.45, 1.5, and 5. $\beta\tau = 10$, $\lambda\tau = 0.3$, $\mu\tau = 25$, $\tau/2 = 0.3$.

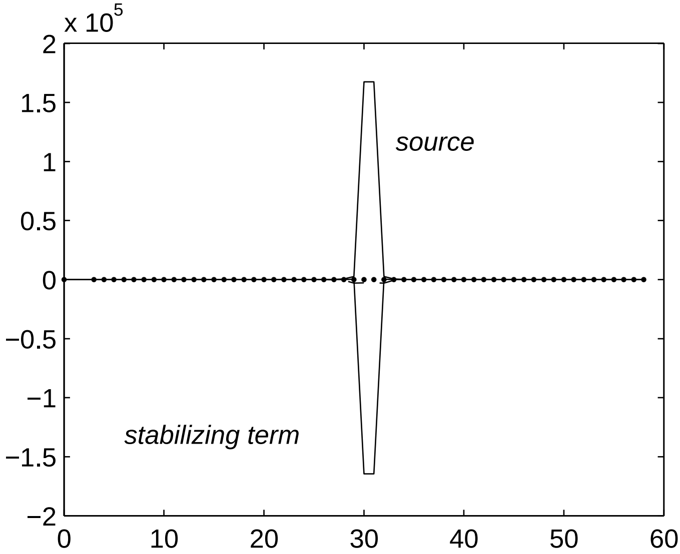

Figure 2: Contribution of different terms.

The platform follows the peak and assumes a convoluted shape. After some transitional period the solution settles as a global structure moving downward with constant velocity. Because of its unsteady character and global extension, such a structure cannot represent the particle at rest.

6 Conclusions

We overcame instability in an extended Hamilton–Jacobi equation for a free particle. Under the proposed extension (12), a stationary smooth soliton is formed representing the particle. The soliton directly addresses the de Broglie's idea of an elementary particle as a result of a continuous nonlinear field theory. Computational experiments demonstrating the formation of the soliton and its stability are presented.

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