An efficient domain-decomposition pseudo-spectral method for solving elliptic differential equations

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SUMMARY

In this paper, a new numerical scheme based on non-overlapping domain decompositions and integrated Chebyshev approximations for solving elliptic differential equations is presented. The distinguishing feature of the present scheme is that it achieves a C_p continuous solution across the interfaces ($p-$the order of the differential equation). Several test problems are employed to verify the method. The obtained results indicate that the achievement of higher-order smoothness leads to a significant improvement in accuracy.

KEY WORDS: domain decomposition; spectral approximation; collocation method; integral formulation

1 INTRODUCTION

Domain decomposition (DD) methods have become necessary to deal with large industrial applications. The methods divides the given analysis domain into a number of subdomains. Based on the concept of spatial decomposition, the DD methods can be classified into two categories: overlapping (Schwarz methods) and non-overlapping (substructuring methods). Continuity of the solution and its smoothness up to a certain order are imposed over contiguous regions. The original problem can thus be replaced with a set of subproblems of reduced size. The main advantages of the DD methods are that they provide an effective way to devise parallel algorithms and to overcome numerical difficulties associated with large matrices and ill-conditioning problems. Furthermore, for spectral methods, the DD methods can be utilized to decompose complex geometries into simple ones where the application of the methods is feasible, and to overcome limitations related to fully populated matrices. The main drawback of the DD methods is that the accuracy of the solution is deteriorated by the fact that the numerical solution is not as smooth as that in the case of single domains. From the literature, the conventional DD methods normally provide a solution that is only up to C_{p-1} continuous over contiguous regions ($p-$the order of the differential equation). It can be seen that the achievement of higher-degrees of continuity is very desirable in the context of domain decomposition. Comprehensive discussions on domain decomposition can be found in, for example, [1,2].

In this paper, non-overlapping domain decompositions are incorporated into pseudo-spectral approximations for solving elliptic differential equations (DEs). A comprehensive review on spectral methods can be found in, for example, [3-5]. Previous findings showed that the use of integration to construct the Chebyshev and radial-basis-function expressions provides an effective way to impose the multiple boundary conditions [6-10]. The present study, which is concerned with the case of domain decomposition, will show that the integral collocation formulation allows a C_p continuous solution, instead of the usual C_{p-1} continuity, across the subdomain interfaces. The present numerical scheme can thus attain an improvement in accuracy over conventional differential formulations. Several numerical examples are included to demonstrate the attractiveness of the present implementation.

The remainder of the paper is organized as follows. In section 2, the integral collocation formulation with Chebyshev polynomials is briefly reviewed. In section 3, the proposed collocation method based on non-overlapping domain decompositions and integrated Chebyshev approximations is presented, where the p-order derivative continuity of the solution is achieved. The method is verified by considering several 1D and 2D problems governed by second- and fourth-order elliptic DEs in section 4. Section 5 gives some concluding remarks.

2 THE INTEGRAL PSEUDO-SPECTRAL FORMULATION

Consider the following DE

$$Lu = f, \quad -1 \leq x \leq 1, \tag{1}$$

where L is a differential operator and f is a given function. This equation is coupled with a set of prescribed boundary conditions to constitute the boundary-value problem.

The domain of interest is represented by a set of unevenly-spaced Gauss-Lobatto (G-L) points

$$\{x_i\}_{i=0}^{N} = \left\{ \cos\left(\frac{\pi i}{N}\right) \right\}_{i=0}^{N}, \tag{2}$$

which cluster at boundaries.

For the integral collocation formulation, one first decomposes the highest-order derivative $d^p u/dx^p$ in the differential equation into the truncated Chebyshev series form

$$\frac{d^p u(x)}{dx^p} = \sum_{k=0}^{N} a_k T_k(x), \tag{3}$$

where $\{a_k\}_{k=0}^{N}$ is the set of expansion coefficients to be found, and $\{T_k(x)\}_{k=0}^{N}$ the set of Chebyshev polynomials of first kind. Expressions for lower derivatives and the variable itself are then obtained through integration as

$$\frac{d^{p-1}u(x)}{dx^{p-1}} = \sum_{k=0}^{N} a_k I_k^{(p-1)}(x) + c_1, \tag{4}$$

$$\frac{d^{p-2}u(x)}{dx^{p-2}} = \sum_{k=0}^{N} a_k I_k^{(p-2)}(x) + c_1 x + c_2, \tag{5}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{du(x)}{dx} = \sum_{k=0}^{N} a_k I_k^{(1)}(x) + c_1 \frac{x^{p-2}}{(p-2)!} + c_2 \frac{x^{p-3}}{(p-3)!} + \cdots c_{p-2}x + c_{p-1}, \tag{6}$$

$$u(x) = \sum_{k=0}^{N} a_k I_k^{(0)}(x) + c_1 \frac{x^{p-1}}{(p-1)!} + c_2 \frac{x^{p-2}}{(p-2)!} + \cdots c_{p-1}x + c_p, \tag{7}$$

where $I_k^{(p-1)}(x) = \int T_k(x)dx$, $I_k^{(p-2)}(x) = \int I_k^{(p-1)}(x)dx, \cdots, I_k^{(0)}(x) = \int I_k^{(1)}(x)dx$, and $c_1, c_2, \cdots, c_p$ are integration constants. It can be seen that apart from the Chebyshev coefficients $\{a_k\}_{k=0}^{N}$, the integral formulation ((3)-(7)) produces new coefficients (integration constants $\{c_i\}_{i=1}^{p}$) whose number is equal to the order of DE/the number of boundary conditions, i.e. p. As a result, it allows one (i) to approximate the DE at the whole set of G-L points and (ii) to add p additional equations to the main system to impose p boundary conditions. Numerical results showed that the integral formulation attains a significant improvement in accuracy and condition number over conventional differential formulations [7].

3 THE PRESENT DOMAIN DECOMPOSITION TECHNIQUE

For the sake of simplicity, the method is presented in detail for the following second-order DE

$$\alpha \frac{d^2u}{dx^2} + \beta \frac{du}{dx} + \gamma u = f(x), \tag{8}$$

defined on the domain $a \leq x \leq b$, subject to the Dirichlet boundary conditions at both ends: $\bar{u}_a$ and $\bar{u}_b$.

The present scheme combines the substructuring method with the integral pseudo-spectral method for solving DEs. The numerical procedure involves two main steps: (i) To find the values of the variable u at the interface points/interior-boundary-points (the interface solution) and (ii) To find the values of the variable u at the interior points in subdomains (the subdomain solution).

3.1 The interface solution

The domain of interest is divided into M subdomains. Each subdomain is discretized using $(N+1)$ G-P points via the following coordinate transformation

$$x^{[j]} = \frac{x_r^{[j]} - x_l^{[j]}}{2}\xi + \frac{x_r^{[j]} + x_l^{[j]}}{2} = \frac{L^{[j]}}{2}\xi + \frac{x_r^{[j]} + x_l^{[j]}}{2}, \tag{9}$$

in which $x_l^{[j]}$ and $x_r^{[j]}$ are the coordinates of the boundary points of a subdomain j, $L^{[j]} = x_r^{[j]} - x_l^{[j]}$, and ξ the G-L points $(-1 \leq \xi \leq 1)$.

The continuity of the solution and its flux leads to the following constraint equations

$$u_N^{[j]} = u_0^{[j+1]}, \tag{10}$$

$$\left(\frac{du}{dx}\right)_N^{[j]} = \left(\frac{du}{dx}\right)_0^{[j+1]}, \tag{11}$$

where $j = \{1, 2, \cdots, M-1\}$.

The present scheme requires the solution u to be continuous, i.e.

$$u_N^{[j]} = u_0^{[j+1]} = \bar{u}_j, \quad j = \{1, 2, \cdots, M-1\}, \tag{12}$$

and its derivatives to be matched at the interfaces. This approach allows an easy implementation (automation) of the computer code.

Consider a subdomain j. Using integrated Chebyshev approximations (3)-(7) with $p=2$, the governing equation (8) and the boundary conditions can be transformed into

$$\frac{4\alpha}{L^{[j]2}}\sum_{k=0}^{N} a_k^{[j]} T_k(\xi) + \frac{2\beta}{L^{[j]}}\left(\sum_{k=0}^{N} a_k^{[j]} I_k^{(1)}(\xi) + c_1^{[j]}\right) + \gamma\left(\sum_{k=0}^{N} a_k^{[j]} I_k^{(0)}(\xi) + c_1^{[j]}\xi + c_2^{[j]}\right) = f(x^{[j]}(\xi)), \tag{13}$$

$$\sum_{k=0}^{N} a_k^{[j]} I_k^{(0)}(-1) - c_1^{[j]} + c_2^{[j]} = \bar{u}_{j-1}, \tag{14}$$

$$\sum_{k=0}^{N} a_k^{[j]} I_k^{(0)}(+1) + c_1^{[j]} + c_2^{[j]} = \bar{u}_j, \tag{15}$$

where $\bar{u}_{j-1} = \bar{u}_a$ for $j=1$, $\bar{u}_j = \bar{u}_b$ for $j=M$, and the unknowns are the set of expansion coefficients and integration constants.

The evaluation of (13) at the whole set of G-L points $\{\xi_i\}_{i=0}^{N}$ plus the boundary conditions (14)-(15) results in a determinate system of equations of the form

$$\mathbf{A}^{[j]} \begin{pmatrix} a_0^{[j]} \\ a_1^{[j]} \\ \cdots \\ a_N^{[j]} \\ c_1^{[j]} \\ c_2^{[j]} \end{pmatrix} = \begin{pmatrix} f_0^{[j]} \\ f_1^{[j]} \\ \cdots \\ f_N^{[j]} \\ \bar{u}_{j-1} \\ \bar{u}_j \end{pmatrix}, \tag{16}$$

or

$$\mathbf{A}^{[j]}\widehat{s}^{[j]} = \begin{pmatrix} \widehat{f}^{[j]} \\ \bar{u}_{j-1} \\ \bar{u}_j \end{pmatrix}, \tag{17}$$

where $\mathbf{A}^{[j]}$ is the known matrix of dimension $(N+3)\times(N+3)$. Unlike conventional differential formulations, the governing equation (8) is forced to be satisfied at the two boundary points exactly in (17) (the first and Nth rows)

$$\alpha\left(\frac{d^2u}{dx^2}\right)_0^{[j]}+\beta\left(\frac{du}{dx}\right)_0^{[j]}+\gamma u_0^{[j]} = f_0^{[j]}, \tag{18}$$

$$\alpha\left(\frac{d^2u}{dx^2}\right)_N^{[j]}+\beta\left(\frac{du}{dx}\right)_N^{[j]}+\gamma u_N^{[j]} = f_N^{[j]}. \tag{19}$$

Solving (17) yields

$$\widehat{s}^{[j]}=\left(\mathbf{A}^{[j]}\right)^{-1}\begin{pmatrix}\widehat{f}^{[j]}\\ \bar{u}_{j-1}\\ \bar{u}_j\end{pmatrix}. \tag{20}$$

As mentioned earlier, the interface unknown vector, namely $(\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_{M-1})^T$, are determined by the imposition of continuity of the first-order normal derivative at the interfaces

$$\left(\frac{du}{dx}\right)_N^{[1]} = \left(\frac{du}{dx}\right)_0^{[2]}, \tag{21}$$

$$\left(\frac{du}{dx}\right)_N^{[2]} = \left(\frac{du}{dx}\right)_0^{[3]}, \tag{22}$$

$$\cdots \qquad \cdots$$

$$\left(\frac{du}{dx}\right)_N^{[M-1]} = \left(\frac{du}{dx}\right)_0^{[M]}, \tag{23}$$

where

$$\frac{du^{[j]}(x(\xi))}{dx}=\frac{2}{L^{[j]}}\left(\sum_{k=0}^{N}a_k^{[j]}I_k^{(1)}(\xi)+c_1^{[j]}+0\right)=\frac{2}{L^{[j]}}\left[I_0^{(1)},I_1^{(1)},\cdots,I_N^{(1)},1,0\right]\widehat{s}^{[j]}. \tag{24}$$

Substituting (20) into (21)-(23) and then imposing the prescribed boundary conditions $\bar{u}_a$ and $\bar{u}_b$ yield the following square system of equations

$$\mathbf{A}_f \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \cdots \\ \bar{u}_{M-1} \end{pmatrix} = \widehat{g}, \tag{25}$$

where $\mathbf{A}_f$ is the known interface matrix of dimension $(M-1)\times(M-1)$, and $\widehat{g}$ the known vector whose components are functions of $f(x)$, $\bar{u}_a$ and $\bar{u}_b$.

From (12), (21)-(23), and (18)-(19), it can be seen that the following relations are imposed at an interface j

$$u_N^{[j]} = u_0^{[j+1]}, \tag{26}$$

$$\left(\frac{du}{dx}\right)_N^{[j]} = \left(\frac{du}{dx}\right)_0^{[j+1]}, \tag{27}$$

$$\alpha\left(\frac{d^2u}{dx^2}\right)_N^{[j]} + \beta\left(\frac{du}{dx}\right)_N^{[j]} + \gamma u_N^{[j]} = f_N^{[j]}, \tag{28}$$

$$\alpha\left(\frac{d^2u}{dx^2}\right)_0^{[j+1]} + \beta\left(\frac{du}{dx}\right)_0^{[j+1]} + \gamma u_0^{[j+1]} = f_0^{[j+1]}. \tag{29}$$

Since $f_N^{[j]} = f_0^{[j+1]}$, (26)-(29) lead to

$$\left(\frac{d^2u}{dx^2}\right)_N^{[j]} = \left(\frac{d^2u}{dx^2}\right)_0^{[j+1]}. \tag{30}$$

Thus, C_p continuity ($p = 2$ in this example) is automatically satisfied in general.

3.2 The subdomain solution

Substitutions of the interface values obtained from solving (25) into (20) yield the sets of expansion coefficients and integration constants for subdomains, and hence the solution to the original problem is obtained. It is noted that each subdomain can be analyzed separately, offering an opportunity for parallelization.

The present numerical scheme can be extended to solve higher-dimensional problems and higher-order ODEs. Similarly, the C_p continuity of the solution over contiguous regions is achieved owing to the satisfaction of the governing equation at the boundary points in each subdomain. For the case of higher-order ODEs, consider an ODE of pth-order (p—an even number). The boundary conditions at the interfaces can be chosen to be $\{u, du/dx, ..., d^{p/2-1}u/dx^{p/2-1}\}$, and these unknown values are then determined by the imposition of continuity in the $(p/2), (p/2+1), \cdots, (p-1)$th-order derivatives across the interfaces. For the case of 2D problems, the integral pseudospectral method for single domains, which was reported previously in [10], is applied here to discretize the governing equation in subdomains.

4 NUMERICAL RESULTS

In the following examples, the accuracy of a numerical solution produced by an approximation scheme is measured via the discrete relative L_2 error defined as

$$N_e = \sqrt{\frac{\sum_{i=0}^{Q} [u_e(x_i) - u(x_i)]^2}{\sum_{i=0}^{Q} u_e(x_i)^2}}, \tag{31}$$

where $(Q+1)$ is the number of test points, and u_e and u are the exact and approximate solutions, respectively. The present formulation is based on point collocation and therefore its implementation is simple. Under certain circumstances (e.g. smooth problems with simple geometries), the present scheme yields spectral accuracy. It is known that the domain-decomposition techniques using finite-difference and finite-element schemes provide only algebraic convergence rates with respect to mesh refinement.

4.1 1D problem

The error N_e is computed using a set of 201 uniformly distributed test points ($Q = 200$), which are generally distinct from the G-L collocation points.

4.1.1 Second-order ODE

Consider the following ODE

$$\frac{d^2u}{dx^2} + \frac{du}{dx} + u = \left[1 - (4\pi)^2\right] \sin(4\pi x) + 4\pi \cos(4\pi x), \tag{32}$$

in the domain $0 \leq x \leq 1$ with the boundary conditions $u(0) = u(1) = 0$. The exact solution can be verified to be

$$u_e = \sin(4\pi x), \tag{33}$$

which is a C^∞ function.

The domain of interest is divided into $2, 3, \cdots, 45$ subdomains, and each subdomain is discretized using 5, 7 and 9 G-L points. In order to show the effect of the achievement of higher-order smoothness on the solution accuracy, the case of a single domain is also considered. Figure 1 shows the accuracy of the integral and differential formulations. It can be seen that the gap between the two curves, representing superior accuracy of the integral formulation over the differential one, for the domain-decomposition case is much wider than that for the single-domain case. This is probably attributable to the fact that the solution is only forced to be C^1 continuous across the interfaces for the differential formulation, but up to C^2 for the integral formulation. For the case of 9 points/subdomain, the condition numbers of $\mathbf{A}_f$ are in the range of $O(10^0) - O(10^2)$ for both formulations.

4.1.2 Fourth-order ODE

Find a function u satisfying the fourth-order ODE

$$\frac{d^4u}{dx^4} + \frac{d^2u}{dx^2} = \left[(4\pi)^2 - 1\right](4\pi)^2 \sin(4\pi x), \tag{34}$$

in $0 \leq x \leq 1$ and the boundary conditions

$$u(0) = 0, \quad \frac{du(0)}{dx} = 4\pi,$$
$$u(1) = 0, \quad \frac{du(1)}{dx} = 4\pi.$$

The exact solution here is also given by (33). A wide range of subdomains from 2 to 36 with an increment of 1 are employed. Each subdomain is represented by 5, 7 and 9 G-L points. Results for N_e obtained by the integral and differential formulations are shown in Figure 2. Again, the relative performance of the integral formulation for the domain-decomposition case is far superior to that for the single-domain case. It indicates that the achievement of higher-order smoothness (C^4 continuous) leads to a significant improvement in accuracy. For the case of 9 points/subdomain, the condition numbers of $\mathbf{A}_f$ are in the range of $O(10^1) - O(10^6)$ for both formulations.

4.2 2D problem

Consider the following Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\left[\cos(\pi x) + \cos(\pi y) + 2\cos(\pi x)\cos(\pi y)\right], \tag{35}$$

in a $L-$shaped domain (Figure 3), subject to the Dirichlet boundary conditions. The exact solution is given by

$$u_e = \frac{1}{\pi^2}\left[1+\cos(\pi x)\right]\left[1+\cos(\pi y)\right], \tag{36}$$

which is plotted in Figure 4. For this problem, it is necessary to decompose the analysis domain into a set of subdomains. Here, three subdomains (Figure 3) with the same tensor product grids are employed, and the differential and integral formulations are applied to discretize the governing equation in each subdomain. The differential formulation provides a C^1 continuous solution across the two interfaces. Owing to the satisfaction of the governing equation at the interface points, the pointwise C^2 continuity is achieved with the integral formulation. Results concerning the error norm of the solution N_e are displayed in Figure 5 using seven grids, namely $3\times 3, 5\times 5, \cdots, 15\times 15$/subdomain, and they indicate that the integral formulation is more accurate than the differential formulation. At $N_t = 341$ ($N_t = NM + 1$: the total number of points), the N_e error of the former is three orders of magnitude better than that of the latter. The two formulations yield the condition number of $\mathbf{A}_f$ in the range of $O(10^0) - O(10^1)$. It can be seen that they both offer an exponential rate of convergence.

The present formulation can be extended to the case of radial basis functions (RBF) in a straightforward manner. However, for single domains, our numerical studies (e.g. [9]) already indicated that integrated RBFs give only an algebraic convergence rate (not exponential one as with Chebyshev polynomials). The advantage of the RBF method is that it does not require an underlying mesh.

5 CONCLUDING REMARKS

This paper presents a new domain-decomposition spectral collocation method for solving elliptic differential equations which achieves a C_p continuous solution across the interfaces, where p is the order of the differential equation. This achievement of higher-order smoothness is due to the use of integration to construct the Chebyshev approximations of the solution in subdomains. Numerical results obtained show that the present numerical scheme alleviates the deterioration of accuracy of the solution caused by structural partitioning.

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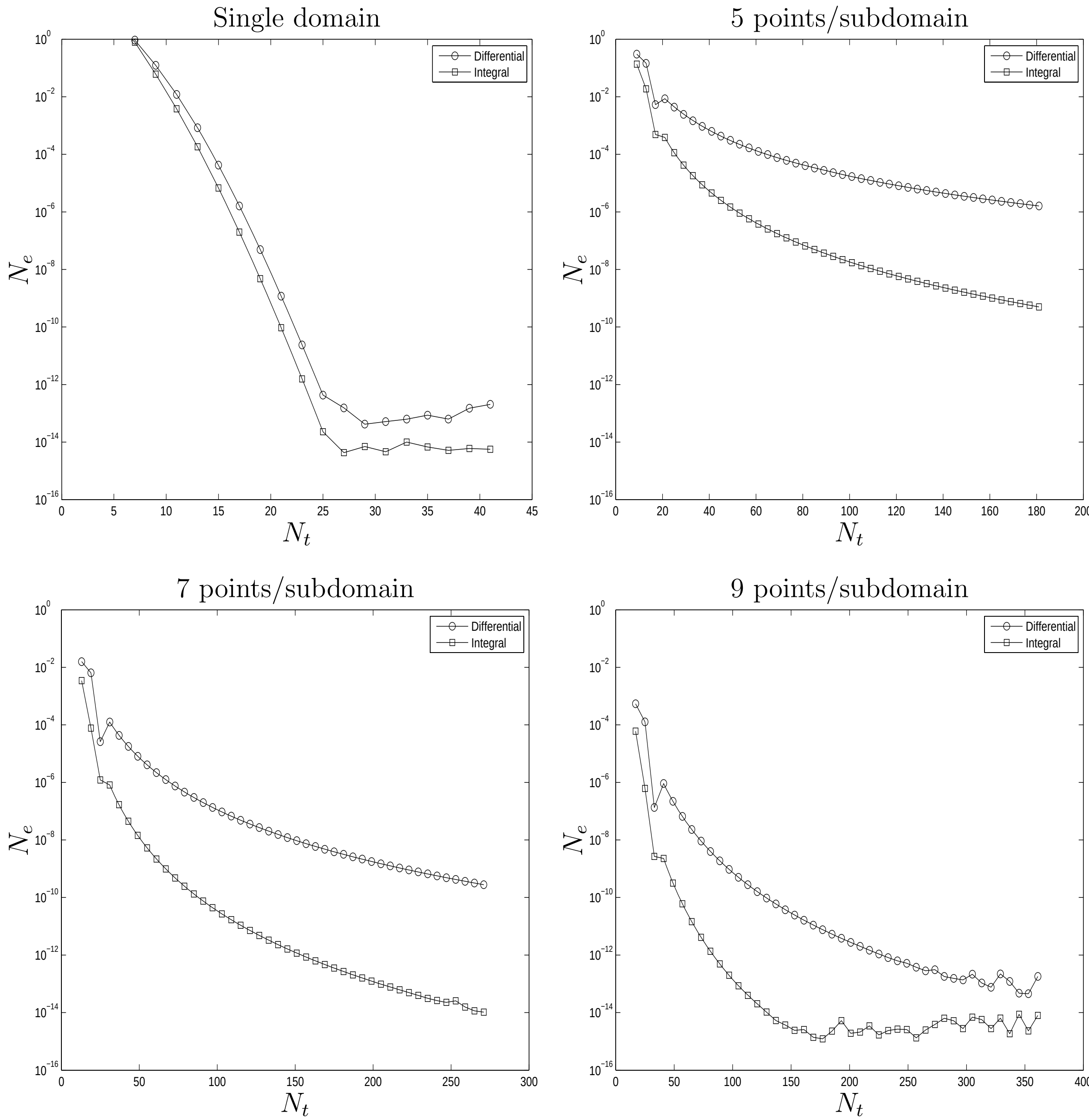

Figure 1: Second-order ODE, $0 \leq x \leq 1$, $\{2, 3, \cdots, 45\}$ subdomains: Discrete relative L_2 error (N_e) versus the total number of points N_t ($N_t = NM + 1$) by the differential and integral formulations. It is noted that all figures have the same scaling for the $y-$axis.

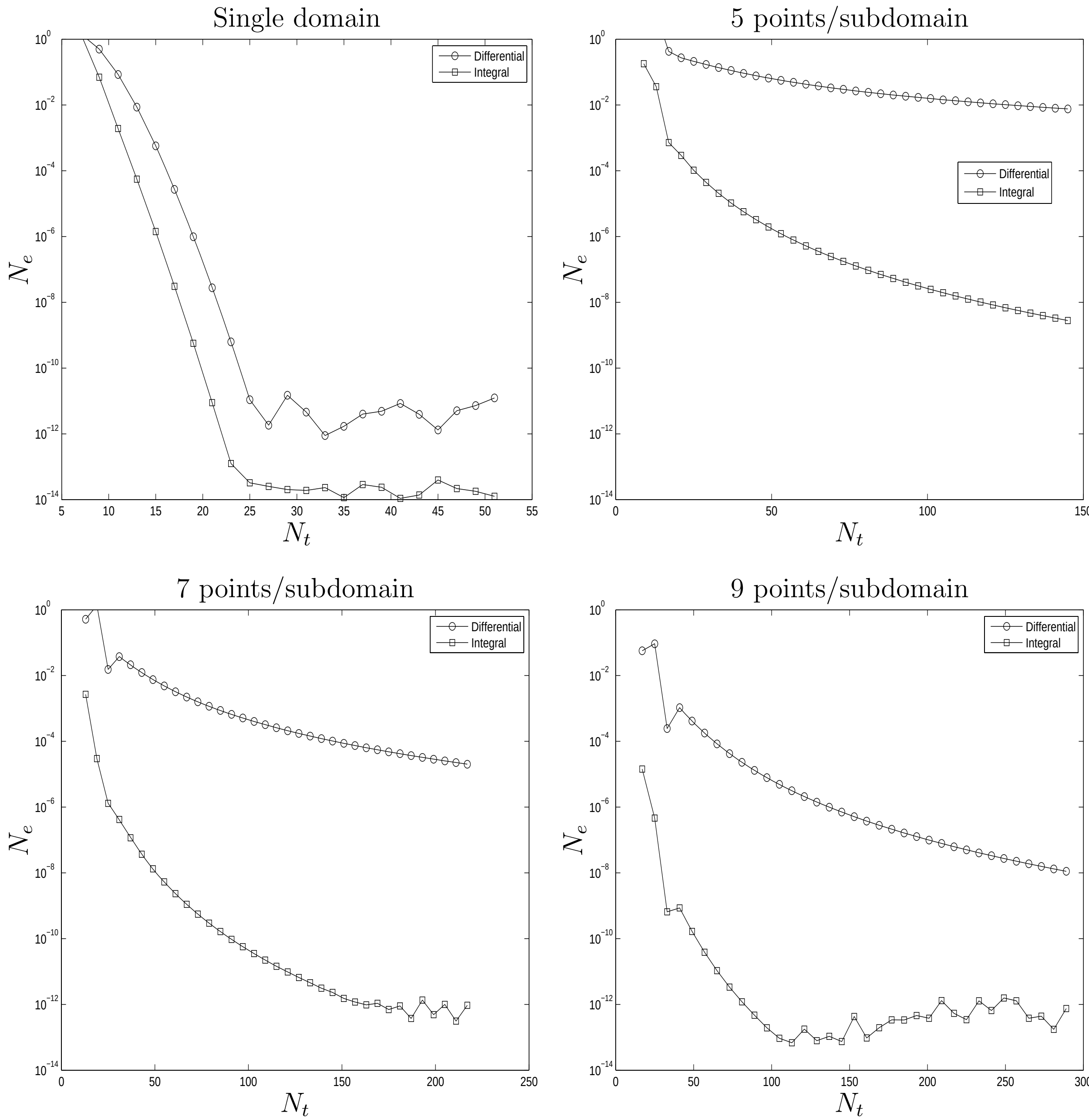

Figure 2: Fourth-order ODE, $0 \leq x \leq 1$, $\{2, 3, \cdots, 36\}$ subdomains: Discrete relative L_2 error (N_e) versus the total number of points (N_t) by the differential and integral formulations. It is noted that all figures have the same scaling for the $y-$axis.

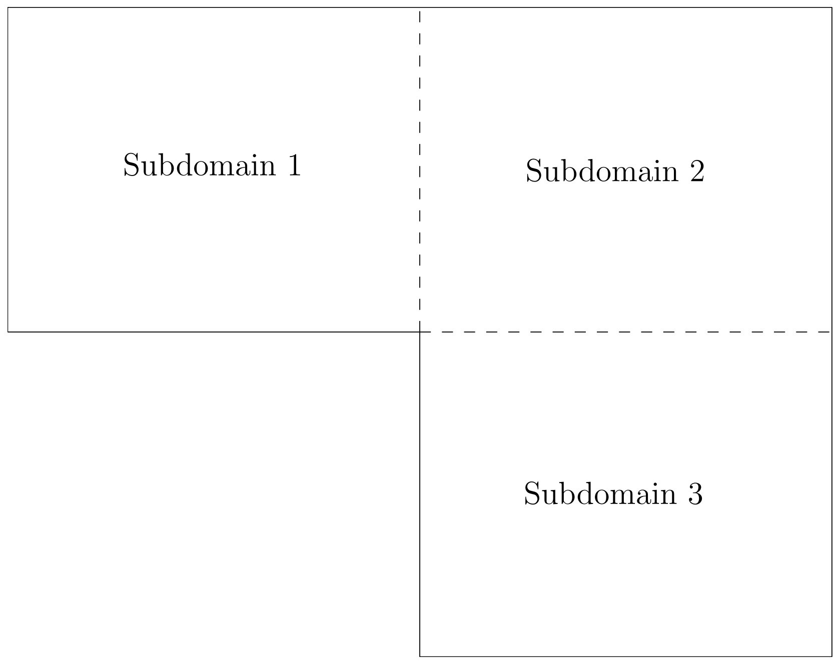

Figure 3: 2D problem: geometry.

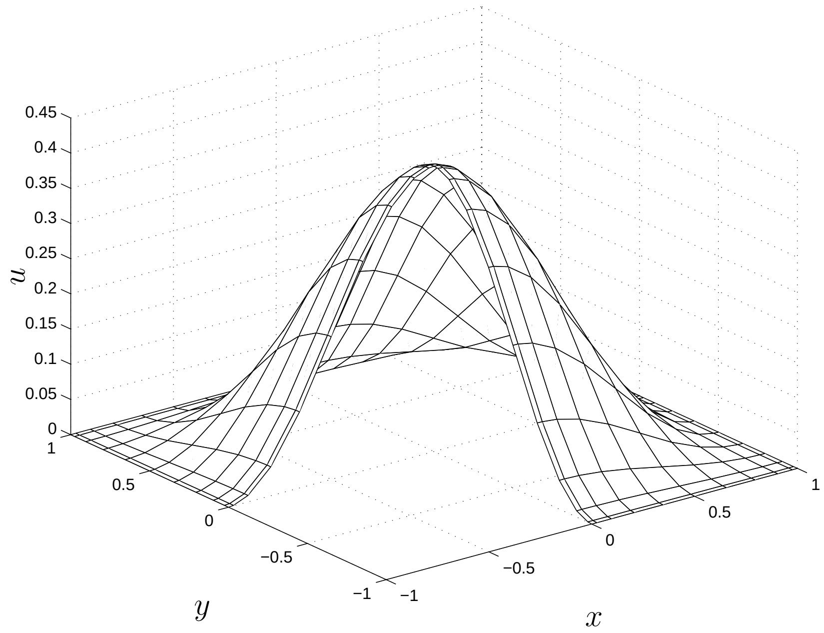

Figure 4: 2D problem: the exact solution.

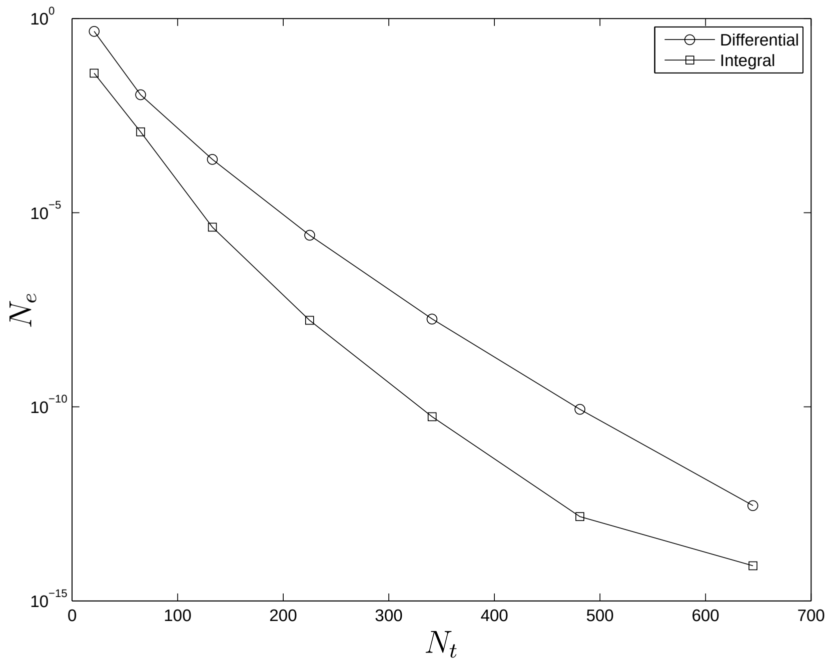

Figure 5: 2D problem, 3 subdomains: Discrete relative L_2 error (N_e) versus the total number of points (N_t) by the differential and integral formulations. At $N_t = 341$, the error of the latter is three orders of magnitude better than that of the former.