OPTIMAL TOPOLOGY SELECTION OF CONTINUUM STRUCTURES WITH STRESS AND DISPLACEMENT CONSTRAINTS

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ABSTRACT: This paper presents three performance indices developed by using the scaling design approach for assisting the selection of optimal topologies for the minimum-weight design of continuum structures subject to stress and displacement constraints. These performance indices are incorporated in the Evolutionary Structural Optimization (ESO) method to monitor the optimization process from which optimal topologies can be identified. Examples provided demonstrate that the proposed performance indices are effective indicators of material efficiency and can be used to compare the efficiency of structural topologies generated by different optimization methods.

KEYWORDS: evolutionary structural optimization, displacement constraint, finite element analysis, performance index, stress constraint, topology

1. INTRODUCTION

The topology optimization of continuum structures has attracted considerable attention in recent years since it offers significant material savings than traditional sizing optimization. The Homogenization method proposed by Bendsøe and Kikuchi [1] can be used to optimize continuum structures by treating element density as design variable. The Soft Kill Option (SKO) method outlined by Mattheck [2] is based on the nature of biological growth. The Evolutionary Structural Optimization (ESO) method presented by Xie and Steven [3], Chu *et al.* [4] and Liang *et al.* [5] has found its full use in engineering practice.

It has been found that using different optimization methods normally results in different topologies for the same problem considered. In addition, even using one method, it is difficult to identify the optimum in the optimization process. As a result of this, performance indices have been attempted by researchers to assist the selection of structural topologies. However, the form factors derived by Burgess [6] are only valid for simple trusses and frames and not applicable to continuum structures. The performance index given by Querin [7] does not consider any type of constraint so that its application is very limited. The indicator of material efficiency presented by Zhao *et al.* [8] is only valid for plane stress structures under a single point load if the displacement constraint is considered.

This paper deals with the optimal topology selection of continuum structures with stress and displacement constraints by using performance indices, which are derived using the scaling design approach. The ESO method for structures with stress and displacement constraints is briefly outlined. Examples are provided to illustrate the capability of the proposed performance indices. Optimal topologies obtained by the present study are compared with those given by other researchers.

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2. FORMULATION OF PERFORMANCE INDICES

The topology optimization problem of a continuum structure can be stated as follows:

$$minimize\ W = \sum_{e=1}^{n} w_e\left(t_e\right) \tag{1}$$

$$subject\ to\ \sigma_{\max}^{VM} - \sigma^* \le 0 \quad or \quad |u_j| - u_j^* \le 0 \qquad j=1, m \tag{2}$$

where W is the total weight of the structure, w_e is the weight of the eth element, t_e is the thickness of the eth element that is also treated as design variable, $\sigma_{\max}^{VM}$ is the maximum von Mises stress of an element in the structure, σ^* is the prescribed stress limit, $|u_j|$ is the magnitude of the $j$the displacement, u_j^* is the prescribed limit of the jth displacement.

To obtain the best feasible design, the element thickness can be scaled at each iteration so that the most active constraint always reaches its limit [9-12]. The stiffness matrix of a plane stress structure is a linear function of the element thickness. For a plane stress structure subject to stress constraints, by scaling the initial design with a factor of $\sigma_{0,\max}^{VM} / \sigma^*$, the scaled weight of the initial design can be expressed by

$$W_0^s = (\sigma_{0,\max}^{VM} / \sigma^*) W_0 \tag{4}$$

where W_0 is the actual weight of the initial design and $\sigma_{0,\max}^{VM}$ is the maximum von Mises stress of an element in the initial design. In a same manner, the scaled weight of the current design at the ith iteration is given as

$$W_i^s = (\sigma_{i,\max}^{VM} / \sigma^*) W_i \tag{5}$$

where W_i is the actual weight of the current design at the ith iteration and $\sigma_{i,\max}^{VM}$ is the maximum von Mises stress of an element in the current design at the ith iteration. The performance index at the ith iteration is defined by

$$PI_s = W_0^s / W_i^s = \sigma_{0,\max}^{VM} W_0 / \sigma_{i,\max}^{vm} W_i \tag{6}$$

For structures with uniformly distributed material density, the performance index can be written as

$$PI_s = \sigma_{0,\max}^{VM} V_0 / \sigma_{i,\max}^{VM} V_i \tag{7}$$

where V_0 and V_i are the volumes of initial design and current design at the ith iteration respectively. For plane stress structures with displacement constraints, the performance index [11] is

$$PI_d = |u_{0j}| V_0 / |u_{ij}| V_i \tag{8}$$

where $|u_{0j}|$ and $|u_{ij}|$ are the most critical constrained displacement in the initial design and in the current design respectively. For plates in bending, the stiffness matrix is the cube root of the plate thickness. The performance index at the ith iteration as presented by Liang *et al.* [12] can be formulated as

$$PI_b = |u_{0j}|^{1/3} V_0 / |u_{ij}|^{1/3} V_i \tag{9}$$

It can be seen from Eqs. 7 to 9 that performance indices are dimensionless numbers which measure the material efficiency in resisting the strength failure or deflection of a structure. They are evaluated by the most active constraint and the volumes at each iteration. Since performance indices are reversely proportional to the volume of the current design, minimizing the weight of a structure can be achieved by maximizing the performance index in an optimization process. It is noted that scaling the element thickness has no effect on the optimal topology or on the performance index, but has a significant effect on the weight of the structure and the active constraint. Hence, the element thickness is not changed in the model at each iteration, but it can be changed in sizing the obtained optimal topology to satisfy the actual prescribed limit.

3. EVOLUTIONARY TOPOLOGY OPTIMIZATION

3.1 STRESS CONSTRAINTS

The ESO method proposed by Xie and Steven [3] is based on the fully stressed design concept, in which lowly stressed elements are systematically removed from the structure to obtain an efficient design. The maximum von Mises stress is used as the element removal criteria, which is expressed by

$$\sigma_e^{VM} < RR_k \sigma_{i,\max}^{VM} \tag{10}$$

in which σ_e^{VM} is the von Mises stress of the eth element, $\sigma_{i,\max}^{VM}$ is the maximum von Mises stress of an element in the current design at the ith iteration and RR_k is the Rejection Ratio at the kth steady state. All elements that satisfy Eq. 10 are deleted from the structure. The cycle of element removal and finite element analysis is repeated by using the same RR_k until no more elements can be deleted from the model at the current steady state. An Evolution Ratio ER is then added to the RR_k, which becomes

$$RR_{k+1} = RR_k + ER \tag{11}$$

Since there is no objective function and constraints involved in the above traditional ESO procedure, the optimal topology for the minimum-weight design cannot be identified during the optimization process. This problem is solved by incorporating the proposed performance index PI_s with stress constraints in the optimization process.

3.2 DISPLACEMENT CONSTRAINTS

In the ESO method for structures with displacement constraints [4], elements with little contribution to the structure stiffness are removed from the structure to achieve the minimum-weight design. Inefficient materials are identified by the sensitivity numbers, which are defined by

$$\alpha_e = |\{u_{ej}\}^T [k_e] \{u_e\}|, \text{ for single displacement constraint} \tag{12}$$

$$\alpha_e = \sum_{j=1}^{m} \lambda_j \, |\{u_{ej}\}^T [k_e] \{u_e\}|, \text{ for multiple displacement constraints} \tag{13}$$

where $\{u_{ej}\}^T$ is the nodal displacement vector of the eth element under the unit load corresponding to the jth displacement component, $[k_e]$ is the stiffness matrix of the eth element, $\{u_e\}$ is the nodal displacement vector of the eth element under the applied loads and the weighting parameter λ_j is chosen as $|u_j|/u_j^*$ and m is the total number of constraints.

Since no objective function is used to control the optimization process in the ESO procedure presented by Chu et al. [4], the optimal topology is difficult to be determined. The performance indices PI_d and PI_b presented can be used in the above ESO procedure to monitor the performance history, from which the optimal topology is easily identified. In the optimization process, only a small number of elements that have the lowest sensitivity numbers are eliminated from the design at each iteration. The Element Removal Ratio (*ERR*) is defined as the ratio of the number of elements to be removed to the total number of elements in the initial design domain.

3.3 OPTIMIZATION PROCEDURE

The evolutionary optimization procedure is given as follows:

Step 1: Model the structure with a fine mesh of finite elements;
Step 2: Analyze the structure for the applied load and unite loads;
Step 3: Calculate the performance index;
Step 4: Calculate the von Mises stress of elements or sensitivity number for each element;
Step 5: Remove elements with low stress level or with the lowest sensitivity numbers;
Step 6: Repeat Steps 2 to 5 until the performance index is less than 1 or constant in later iterations.

4. EXAMPLES

4.1 EXAMPLE 1

The ESO method for structures with stress constraints is used to optimize the beam with fixed ends as shown in Fig. 1. A concentrated load of 100 kN is applied to the top of the mid span of the beam. The design domain is distretized into 90x30 four-node plane stress elements. The Young's modulus of material *E*=200 GPa, Poisson's ratio *ν*=0.3 and element thickness *t*=15 mm are assumed. The Rejection Ratio $RR_0 = 1\%$ and Evolution Ratio *ER*=1% are adopted in the optimization process. The performance index history of the beam is shown in Fig. 2. It can be seen that the maximum performance index is 8.54, which means that the scaled weight of the initial design is 8.54 times that the optimal topology as illustrated in Fig. 3 while the maximum von Mises stress of elements in the design reaches the limit. Fig. 4 shows the regenerated final design proposal given by Mattheck [2] using the Soft Kill Option (SKO) method. The performance index of this proposal by using Eq. 7 is found to be 1.51, which is much less than that obtained by the ESO method.

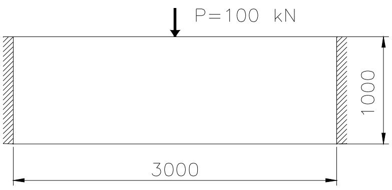

Fig. 1 Design domain

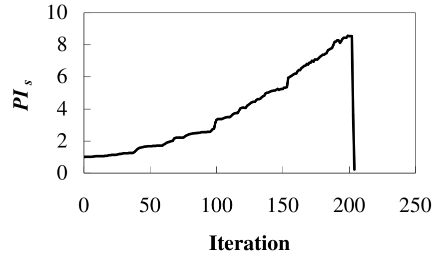

Fig. 2 Performance index history

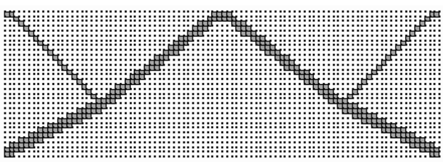

Fig. 3 Optimal topology

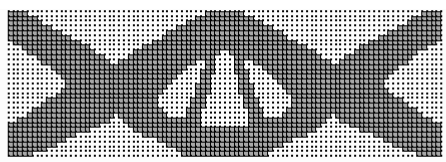

Fig. 4 Design proposal by Mattheck [2]

4.2 EXAMPLE 2

The design domain of a cantilever beams shown in Fig. 5 is modeled using 32x20 four-node plane stress elements. A concentrated load of 3 kN is placed at the centre of the free end where a displacement constraint is imposed. The material properties E=200 GPa, v=0.3 and t=1mm are used. The element removal ratio ERR=2% is employed in this case. The performance index history is shown in Fig.6, from which it is seen that the maximum performance index is 1.20 for the optimal topology presented in Fig. 7. The performance index of the topology obtained by Chu *et al.* [4] using the ESO method as illustrated in Fig. 8 is 1.11, which is calculated by Eq. 8. The topology shown in Fig. 9 is presented by Suzuki and Kikuchi [13] using the Homogenization method and its performance index is found to be 1.04. It is clear that the optimal topology given by the present study has a higher efficiency than those obtained by other researchers.

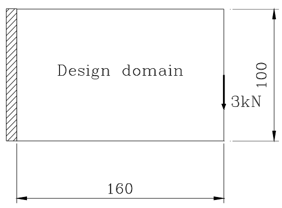

Fig. 5 Design domain

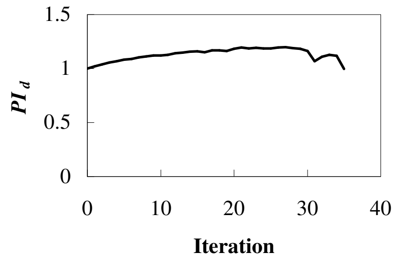

Fig. 6 Performance index history

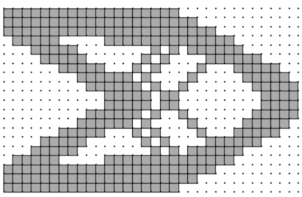

Fig. 7 Topology by present study

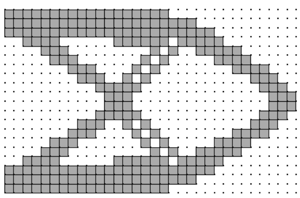

Fig. 8 Topology by Chu et al. [4]

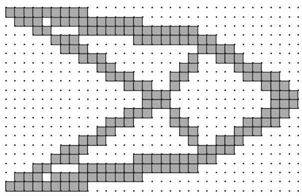

Fig. 9 Topology by Suzuki and Kikuchi [13]

4.3 EXAMPLE 3

A simply supported square plate (200x200) under a point load of 0.04 N at the centre is optimized using the performance index PI_b with the ESO method. The initial plate is modeled using 6400 three-node plate elements. A displacement constraint is imposed at the centre of the plate. E=174.72 GPa, v=0.3 and t=0.1 mm are adopted. ERR=1% is used in the optimization process. It is seen from Fig. 10 that the maximum performance index obtained by the present study is 1.64, which corresponds to the topology illustrated in Fig. 11. The performance index of the topology given by Atrek [14] as shown in Fig. 12 is 1.61. It can be concluded that the efficiency of material layout in the bending plate can be compared via the proposed performance index PI_b.

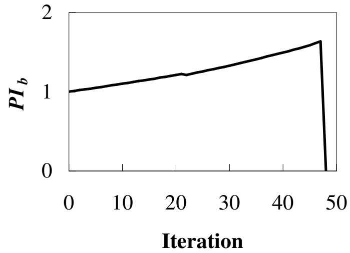

Fig. 10 Performance index history

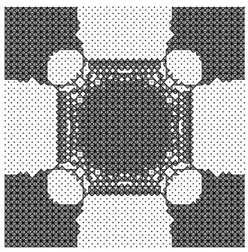

Fig. 11 Topology by present study

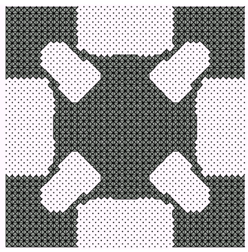

Fig. 12 Topology by Atrek [14]

5. CONCLUSIONS

Three performance indices have been presented for the topology optimization of plane stress structures with stress and displacement constraints and of plates in bending. It is shown that the proposed performance index can be incorporated in any structural optimization method such as the ESO approach to monitor the optimization process, from which the optimal topology of the structure can be easily identified. In addition, the efficiency of structural topologies produced by different optimization methods can be objectively evaluated by using the performance indices.

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