RBF interpolation of boundary values in the BEM for heat transfer problems

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Abstract This paper is concerned with the application of radial basis function networks (RBFNs) as interpolation functions for all boundary values in the boundary element method (BEM) for the numerical solution of heat transfer problems. The quality of the estimate of boundary integrals is greatly affected by the type of functions used to interpolate the temperature, its normal derivative and the geometry along the boundary from the nodal values. In this paper, instead of conventional Lagrange polynomials, interpolation functions representing these variables are based on the “universal approximator” RBFNs, resulting in much better estimates. The proposed method is verified on problems with different variations of temperature on the boundary from linear level to higher orders. Numerical results obtained show that the BEM with indirect RBFN (IRBFN) interpolation performs much better than the one with linear or quadratic elements in terms of accuracy and convergence rate. For example, for the solution of Laplace’s equation in 2D, the BEM can achieve the norm of error of the boundary solution of $O(10^{-5})$ by using IRBFN interpolation while quadratic BEM can achieve a norm only of $O(10^{-2})$ with the same boundary points employed. The IRBFN-BEM also appears to have achieved a higher efficiency. Furthermore, the convergence rates are of $O(h^{1.38})$ and $O(h^{4.78})$ for the quadratic BEM and the IRBFN-based BEM, respectively, where $h$ is the nodal spacing.

1. Introduction

Boundary element methods (BEMs) have become one of the popular techniques for solving boundary value problems in continuum mechanics. For linear homogeneous problems, the solution procedure of BEM consists of two main stages:

1. estimate the boundary solution by solving boundary integral equations (BIEs), and

2. estimate the internal solution by calculating the boundary integrals (BIs) using the results obtained from the stage (1).

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The first stage plays an important role, because the solution obtained here provides sources to compute the internal solution. However, it can be seen that both stages involve the evaluation of BIs, of which any improvements achieved result in the betterment of the overall solution to the problem. In the evaluation of BIs, the two main topics of interest are how to represent the variables along the boundary adequately and how to evaluate the integrals accurately, especially in the cases where the moving field point coincides with the source point (singular integrals). In the standard BEM (Banerjee and Butterfield, 1981; Brebbia et al., 1984), the boundary of the domain of analysis is divided into a number of small segments (elements). The geometry of an element and the variation of temperature and temperature gradient over such an element are usually represented by Lagrange polynomials, of which the constant, linear and quadratic types are the most widely applied. With regard to the evaluation of integrals, including weakly and strongly singular integrals, considerable achievements have been reported by Sladek and Sladek (1998). It is observed that the accuracy of solution by the standard BEM greatly depends on the type of elements used. On the other hand, neural networks (NN) which deal with interpolation and approximation of functions, have been developed recently and become one of the main fields of research in numerical analysis (Haykin, 1999). It has been proved that the NNs are capable of universal approximation (Cybenko, 1989; Girosi and Poggio, 1990). Interest in the application of NNs (especially the multi quadricle (MQ) radial basis function networks (RBFNs)) for numerical solution of PDEs has been increasing (Kansa, 1990; Mai-Duy and Tran-Cong, 2001a, b, 2002; Sharan et al., 1997; Zerroukat et al., 1998). In this study, “universal approximator” RBFNs are introduced into the BEM scheme to represent the variables along the boundary. Although RBFNs have an ability to represent any continuous function to a prescribed degree of accuracy, practical means to acquire sufficient approximation accuracy still remain an open problem. Indirect RBFNs (IRBFNs) which perform better than direct RBFNs in terms of accuracy and convergence rate (Mai-Duy and Tran-Cong, 2001a, 2002) are utilised in this work. Due to the presence of NNs in BIs, the treatment of the singularity in CPV integrals requires some modification in comparison with the standard BEM. The paper is organised as follows. In Section 2, the IRBFN interpolation of functions is presented and its performance is then compared with linear and quadratic element results via a numerical example. Section 3 is to introduce the IRBFN interpolation into the BEM scheme to represent the variable in BIEs. In Section 4, some 2D heat transfer problems governed by Laplace’s or Poisson’s equations are simulated to validate the proposed method. Section 5 gives some concluding remarks.

2. Interpolation with IRBFN

The task of interpolation problems is to estimate a function $y(s)$ for arbitrary $s$ from the known value of $y(s)$ at a set of points $s^{(1)}, s^{(2)}, \ldots, s^{(n)}$ and therefore,
the interpolation must model the function by some plausible functional form. The form is expected to be sufficiently general in order to describe large classes of functions which might arise in practice. By far the most common functional forms used are based on polynomials (Press et al., 1988). Generally, for problems of interpolation, universal approximators are highly desired in order to handle large classes of functions. It has been proved that RBFNs, which can be considered as approximation schemes, are able to approximate arbitrarily well continuous functions (Girosi and Poggio, 1990). The function \( y \) to be interpolated/approximated is decomposed into radial basis functions as

\[
y(x) \approx f(x) = \sum_{i=1}^{m} w^{(i)} g^{(i)}(x),
\]

where \( m \) is the number of radial basis functions, \( \{g^{(i)}\}_{i=1}^{m} \) is the set of chosen radial basis functions and \( \{w^{(i)}\}_{i=1}^{m} \) is the set of weights to be found. Theoretically, the larger the number of radial basis functions used, the more accurate the approximation will be as, stated in Cover’s theorem (Haykin, 1999). However, the difficulty here is how to choose the network’s parameters such as RBF widths properly. IRBFNs were found to be more accurate than direct RBFNs with relatively easier choice of RBF widths (Mai-Duy and Tran-Cong, 2001a, 2002) and will be employed in the present work. In this paper, only the problems in 2D are discussed. In view of the fact that the interpolation IRBFN method will be coupled later with the BEM where the problem dimensionality is reduced by one, only the MQ-IRBFN for function and its derivatives (e.g. up to the second order) in 1D needs to be employed here and its formulation is briefly recaptured as follows:

\[
y''(s) \approx f''(s) = \sum_{i=1}^{m} w^{(i)} g^{(i)}(s),
\]

\[
y'(s) \approx f'(s) = \sum_{i=1}^{m} w^{(i)} H^{(i)}(s) + C_1,
\]

\[
y(s) \approx f(s) = \sum_{i=1}^{m} w^{(i)} \tilde{H}^{(i)}(s) + C_1 s + C_2,
\]

where \( s \) is the curvilinear coordinate (arc-length), \( C_1 \) and \( C_2 \) are constants of integration and

\[
g^{(i)}(s) = ((s - c^{(i)})^2 + a^{(i)^2})^{1/2},
\]
\[ H^{(i)}(s) = \int g^{(i)}(s) \, ds = \frac{(s - c^{(i)})(s - c^{(i)})^2 + a^{(i)2})^{1/2}}{2} \]

\[ + \frac{a^{(i)2}}{2} \ln((s - c^{(i)}) + ((s - c^{(i)})^2 + a^{(i)2})^{1/2}), \]

(6)

\[ H^{(i)}(s) = \int H^{(i)}(s) \, ds = \frac{((s - c^{(i)})^2 + a^{(i)2})^{3/2}}{6} \]

\[ + \frac{a^{(i)2}}{2}(s - c^{(i)})\ln((s - c^{(i)}) + ((s - c^{(i)})^2 + a^{(i)2})^{1/2}) \]

\[ - \frac{a^{(i)2}}{2}((s - c^{(i)})^2 + a^{(i)2})^{1/2}, \]

(7)

in which \( \{c^{(i)}\}_{i=1}^{m} \) is the set of centres and \( \{a^{(i)}\}_{i=1}^{m} \) is the set of RBF widths. The RBF width is chosen based on the following simple relation

\[ a^{(i)} = \beta d^{(i)}, \]

where \( \beta \) is a factor and \( d^{(i)} \) is the minimum arclength between the \( i \)th centre and its neighbouring centres. Since \( C_1 \) and \( C_2 \) are to be found, it is convenient to let \( w^{(m+1)} = C_1, \ w^{(m+2)} = C_2, \ \bar{H}^{(m+1)} = s \) and \( \bar{H}^{(m+2)} = 1 \) in equation (4), which becomes

\[ y(s) \approx f(s) = \sum_{i=1}^{m+2} w^{(i)} \bar{H}^{(i)}(s), \]

(8)

\[ \bar{H}^{(i)} = \text{RHS of equation (7), } i = 1, \ldots, m, \]

(9)

\[ \bar{H}^{(m+1)} = s, \]

(10)

\[ \bar{H}^{(m+2)} = 1. \]

(11)

The detailed implementation and accuracy of the IRBFN method were reported previously (Mai-Duy and Tran-Cong, 2002). In all the numerical examples carried out in this paper, the value of \( \beta \) is simply chosen to be in the range of 7-10. Before introducing the IRBFN interpolation into the BEM scheme, the performance of the IRBFN and element-based method are compared using the interpolation of the following function
\[ y = 0.02(12 + 3s - 3.5s^2 + 7.2s^3)(1 + \cos 4\pi s)(1 + 0.8 \sin 3\pi s), \]
where \(0 \leq s \leq 1\) (Figure 1). The accuracy achieved by each technique is evaluated via the norm of relative error of the solution \(N_e\) defined by

\[ N_e = \left( \frac{\sum_{i=1}^{q} (y(s^{(i)}) - f(s^{(i)}))^2}{\sum_{i=1}^{q} y(s^{(i)})^2} \right)^{1/2}, \]  

where \(y(s^{(i)})\) and \(f(s^{(i)})\) are the exact and approximate solutions at the point \(i\), respectively, and \(q\) is the number of test points. The performance of linear, quadratic and IRBFN interpolations are assessed using four data sets of 13, 15, 17 and 19 known points. For each data set, the function \(y\) is estimated at 500 test points. Note that the known and test points here are uniformly distributed. The results obtained using \(\beta = 10\) are displayed in Figure 2 showing that the IRBFN method achieves superior accuracy and convergence rate to the element-based method. The solution converges apparently as \(O(h^{1.95})\), \(O(h^{1.98})\) and \(O(h^{9.47})\) for linear, quadratic and IRBFN interpolations, respectively, where \(h\) is the grid point spacing. At \(h = 0.06\), which corresponds to a set of 19 grid points.

![Figure 1. Interpolation of function](image-url)
For heat transfer problems, the governing equations take the form

$$\nabla^2 u = b, \quad \mathbf{x} \in \Omega, \tag{13}$$

$$u = \bar{u}, \quad \mathbf{x} \in \Gamma_u, \tag{14}$$

$$q \equiv \frac{\partial u}{\partial n} = \bar{q}, \quad \mathbf{x} \in \Gamma_q, \tag{15}$$

where $u$ is the temperature, $q$ is the temperature gradient across the surface, $\mathbf{n}$ is the unit outward normal vector, $\bar{u}$ and $\bar{q}$ are the prescribed boundary conditions, $b$ is a known function of position and $\Gamma = \Gamma_u + \Gamma_q$ is the boundary of the domain $\Omega$.

Integral equation (IE) formulations for heat transfer problems are well documented in a number of texts (Banerjee and Butterfield, 1981; Brebbia et al., 1984). Equations (13)-(15) can be reformulated in terms of the IEs for a given spatial point $\xi$ as follows.
\[ c(\xi)u(\xi) + \int_{\Gamma} q^*(\xi, x)u(x) \, d\Gamma + \int_{\Omega} b(x)u^*(\xi, x) \, d\Omega = \int_{\Gamma} u^*(\xi, x)q(x) \, d\Gamma, \]  

(16)

where \( u^* \) is the fundamental solution to the Laplace equation, e.g. for a 2D isotropic domain \( u^* = (1/2\pi)\ln(1/r) \) in which \( r \) is the distance from the point \( \xi \) to the current point of integration \( x \), \( q^* = \partial u^* / \partial n \), \( c(\xi) = \theta/2\pi \) with \( \theta \) being the internal angle of the corner in radians, if \( \xi \) is a boundary point and \( c(\xi) = 1 \), if \( \xi \) is an internal point. Note that the volume integral here does not introduce any unknowns because the function \( b \) is given and furthermore, it can be reduced to the BIs by using the particular solution (PS) techniques (Zheng et al., 1991) or the dual reciprocity method (DRM) (Partridge et al., 1992). Without loss of generality, the following discussions are based on equation (16) with \( b = 0 \) (Laplace’s equation).

For the standard BEM, the numerical procedure for equation (16) involves a subdivision of the boundary \( \Gamma \) into a number of small elements. On each element, the geometry and the variation of \( u \) and \( q \) are assumed to have a certain shape such as linear and quadratic ones. The study on the interpolation of function in Section 2 shows that the IRBFN interpolation achieves an accuracy and convergence rate superior to the linear and quadratic element-based interpolations. The question here is whether the employment of IRBFN interpolation in the BEM scheme can improve the solution in terms of accuracy and convergence rate as in the case of function approximation. The answer is positive and substantiated in the remainder of this paper.

The first issue to be considered is about the implementation of singular integrals when IRBFNs are present within integrands. The difference between the IRBFN and the Lagrange-type interpolation is that in the present IRBFN interpolation, none of the basis functions are null at the singular point (the point where the field point \( x \) and the source point \( \xi \) coincide) and hence the corresponding integrands obtained are not regular. Consequently, at the singular point all CPV integrals associated with the IRBFN weights are singular and cannot be evaluated by using the hypothesis of constant potential directly over the whole domain as in the case of the standard BEM. To overcome this difficulty, the treatment of singular CPV integrals needs to be slightly modified. The BIEs can be written in the following form (Hwang et al., 2002; Tanaka et al., 1994)

\[ u(\xi) \int_{\Gamma_{\epsilon}} q^*(\xi, x) \, d\Gamma + \text{CPV} \int_{\Gamma} q^*(\xi, x)u(x) \, d\Gamma = \int_{\Gamma} u^*(\xi, x)q(x) \, d\Gamma, \]  

(17)

where \( \Gamma_{\epsilon} \) is part of a circle that excludes its origin (or the singular point) from the domain of analysis. Assume that the temperature \( u(x) \) is a constant unit on
the whole domain, i.e. \( u(\xi) = u(x) = 1 \), and hence the gradient \( q(x) \) is everywhere zero. Equation (17) then simplifies to

\[
\int_{\Gamma_{\xi \rightarrow 0}} q^*(\xi, x) \, d\Gamma = -\text{CPV} \int_{\Gamma} q^*(\xi, x) \, d\Gamma. \tag{18}
\]

Substitution of equation (18) into equation (17) yields

\[
\text{CPV} \int_{\Gamma} q^*(\xi, x)(u(x) - u(\xi)) \, d\Gamma = \int_{\Gamma} u^*(\xi, x)q(x) \, d\Gamma. \tag{19}
\]

The CPV integral is now written in the non-singular form, where the standard Gaussian quadrature can be applied. For weakly singular integrals, some well-known treatments such as logarithmic Gaussian quadrature and Telles’ transformation technique (Telles, 1987) can be applied directly as in the case of the standard BEM.

The second issue is concerned with the employment of the IRBFNs in the BEM scheme to represent the variables in the BIs. In the present method, the boundary \( \Gamma \) of the domain of analysis is also divided into a number of segments \( N_s \), i.e.

\[
\Gamma = \sum_{j=1}^{N_s} \Gamma_j,
\]

which are 1D domains to be represented by networks. Note that the size of the segment \( \Gamma_j \) can be much larger than the size of elements in the standard BEM provided that the associated boundary is smooth and the prescribed boundary conditions are of the same type. Equation (19) can be written in the discretised form as

\[
\sum_{j=1}^{N_s} \int_{\Gamma_j} q^*(\xi, x)(u_j(x) - u_l(\xi)) \, d\Gamma_j = \sum_{j=1}^{N_s} \int_{\Gamma_j} u^*(\xi, x)q_j(x) \, d\Gamma_j, \tag{20}
\]

where the subscript \( j \) denotes the general segments and the subscript \( l \) indicates the segment containing the source point \( \xi \). The variation of temperature \( u \) and gradient \( q \) on the segment \( \Gamma_j \) is now represented by the IRBFNs in terms of the curvilinear coordinate \( s \) as (equation (9))

\[
u_j = \sum_{i=1}^{n_j+2} w_{ij} H_j^{(i)}(s), \tag{21}\]
\[ q_j = \sum_{i=1}^{mj+2} w_{qj}^{(i)} \tilde{H}_j^{(i)}(s), \]  

(22) \hfill \text{RBF interpolation of boundary values}

where \( s \in \Gamma_j \), \( mj + 2 \) is the number of IRBFN weights, \( \{w_{uj}^{(i)}\}_{i=1}^{mj+2} \) and \( \{w_{qj}^{(i)}\}_{i=1}^{mj+2} \) are the sets of weights of networks for the temperature \( u \) and temperature gradient \( q \), respectively. Similarly, the geometry can be interpolated from the nodal value by using the IRBFNs as

\[ x_{1j} = \sum_{i=1}^{mj+2} w_{x_{1j}}^{(i)} \tilde{H}_j^{(i)}(s), \]  

(23)

\[ x_{2j} = \sum_{i=1}^{mj+2} w_{x_{2j}}^{(i)} \tilde{H}_j^{(i)}(s). \]  

(24)

Substitution of equations (21) and (22) into equation (20) yields

\[ \sum_{j=1}^{Ns} \int_{\Gamma_j} q^*(\xi, s) \left( \sum_{i=1}^{mj+2} w_{uj}^{(i)} \tilde{H}_j^{(i)}(s) - \sum_{i=1}^{ml+2} w_{ul}^{(i)} \tilde{H}_l^{(i)}(\xi) \right) d\Gamma_j \]

(25)

\[ = \sum_{j=1}^{Ns} \int_{\Gamma_j} u^*(\xi, s) \left( \sum_{i=1}^{mj+2} w_{qj}^{(i)} \tilde{H}_j^{(i)}(s) \right) d\Gamma_j, \]

or,

\[ \sum_{j=1}^{Ns} \left\{ \sum_{i=1}^{mj+2} w_{uj}^{(i)} \left( \int_{\Gamma_j} q^*(\xi, s) \tilde{H}_j^{(i)}(s) d\Gamma_j \right) - \sum_{i=1}^{ml+2} w_{ul}^{(i)} \left( \int_{\Gamma_j} q^*(\xi, s) \tilde{H}_l^{(i)}(s) d\Gamma_j \right) \right\} \]

(26)

\[ = \sum_{j=1}^{Ns} \sum_{i=1}^{mj+2} w_{qj}^{(i)} \left( \int_{\Gamma_j} u^*(\xi, s) \tilde{H}_j^{(i)}(s) d\Gamma_j \right), \]

where \( mj \) is the number of training points on the segment \( j \), which can vary from segment to segment. Equation (26) is formulated in terms of the IRBFN weights of networks for \( u \) and \( q \) rather than the nodal values of \( u \) and \( q \) as in the case of the standard BEM. Locating the source point \( \xi \) at the boundary training points results in the underdetermined system of algebraic equations with the unknown being the IRBFN weights. Thus, the system of equations obtained, which can have many solutions, needs to be solved in the general least squares sense. The preferred solution is the one whose values are smallest in the least squares sense (i.e. the norm of components is minimum). This can be achieved by using singular value decomposition technique (SVD). The procedural flowchart can be briefly summarised as follows:
(1) divide the boundary into a number of segments over each of which the boundary is smooth and the prescribed boundary conditions are of the same type;

(2) apply the IRBFN for approximation of the prescribed physical boundary conditions in order to obtain the IRBFN weights which are the boundary conditions in the weight space;

(3) form the system matrices associated with the IRBFN weights $w_u$ and $w_q$;

(4) impose the boundary conditions obtained from the step 2 and then solve the system for IRBFN weights by the SVD technique;

(5) compute the boundary solution by using the IRBFN interpolation;

(6) evaluate the temperature and its derivatives at selected internal points;

(7) output the results.

Note that for the numerical solution of Poisson's equations using the BEM-PS approach, the PS is first found by expressing the known function $b$ as a linear combination of radial basis functions and the volume integral is then transformed into the BIs (Zheng et al., 1991). However, the first stage of this process produces a certain error which is separate from the error in the evaluation of the BIs. In order to confine the error of solution only to the evaluation of BIs, the following numerical examples of heat transfer problems governed by the Laplace's equations or Poisson's equations are chosen where the associated analytical PSs exist for the latter.

4. Numerical examples

In this section, the proposed method is verified and compared with the standard BEM on heat transfer problems governed by the Laplace's or Poisson's equations. In order to make the BEM programs general in the sense that they can deal with any types of boundary conditions at the corners, all BEM codes with linear, quadratic and IRBFN interpolations employ discontinuous elements at the corner. The extreme boundary point at the corner is shifted into the element by one-fourth of the length of the element. Integrals are evaluated by using the standard Gaussian quadrature for regular cases and logarithmic Gaussian quadrature or Telles' quadratic transformation (Telles, 1987) for weakly singular cases with nine integration points. For the purpose of error estimation and convergence study, the error norm defined in equation (12) will be utilised here with the function $y$ being the temperature $u$ and its normal derivative $q$ in the case of the boundary solution or the temperature $u$ in the case of the internal solution.

4.1 Boundary geometry with straight lines

It can be seen that the linear interpolation is able to represent exactly the geometry for a straight line and hence on the straight line segment the IRBFN
interpolation needs only to be used for representing the variation of temperature and gradient.

4.1.1 Example 1. Consider a square closed domain whose dimensions are taken to be 6 by 6 units as shown in Figure 3. The temperature on the left and right edges is maintained at 300 and 0, respectively, while the homogeneous Neumann conditions \( q = 0 \) are imposed on the other edges. Inside the square, the steady-state temperature satisfies the Laplace’s equation. The analytical solution is

\[
    u(x_1, x_2) = 300 - 50x_1.
\]

This is a simple problem where the variation of temperature is linear. It can be seen that the use of linear interpolation is the best choice for this problem. Both linear and IRBFN (\( \beta = 10 \)) interpolations are employed and the corresponding BEM results on the boundary and at some internal points are displayed in Table I showing that the proposed method as well as the linear-BEM works. Significantly, the IRBFN-BEM works increasingly better than the linear-BEM as the number of boundary points increases, which seems to indicate that the IRBFN-BEM does not suffer numerical ill-conditioning as in the case of the standard BEM. Note that in the case of the IRBFN interpolation, each edge of the square domain and the boundary points on it become the domain and training points of the network associated with the edge, respectively. It is expected that the IRBFN-BEM approach performs better in dealing with higher order variations of temperature, which is verified in the following examples.
4.1.2 Example 2. The problem is to find the temperature field such that

\[ \nabla^2 u = 0 \quad \text{inside the square } 0 \leq x_1 \leq \pi, \ 0 \leq x_2 \leq \pi, \quad (27) \]

\[ u(x_1, \pi) = \sin(x_1) \quad \text{on the top edge } (0 \leq x_1 \leq \pi), \quad (28) \]

\[ u(x_1, x_2) = 0 \quad \text{on the other three sides.} \quad (29) \]

The exact solution of this problem is given by Snider (1999)

\[ u(x_1, x_2) = \frac{1}{\sinh(\pi)} \sin(x_1) \sinh(x_2). \]

This is a Dirichlet problem for which the essential boundary condition is imposed along the boundary. Using discontinuous boundary elements at the corner for the case of the standard BEM or shifting the training points at the corner into the adjacent segments for the case of the IRBFN-BEM allows the correct description of multi-valued gradient \( q \) at the corner. In the case of IRBFN interpolation, each side of the square domain becomes the domain of network and the boundary points on it are utilized as training points. To study the convergence of the present method, four boundary point densities, namely \( 5 \times 4, 7 \times 4, 9 \times 4 \) and \( 11 \times 4 \), and \( \beta = 7 \) are employed. Some internal points are selected at \( (\pi/3, \pi/3), (\pi/3, 2\pi/3), (\pi/2, \pi/2), (2\pi/3, \pi/3) \) and \( (2\pi/3, 2\pi/3) \). The performance of the BEM with linear, quadratic and IRBFN interpolations is assessed using the error norms of the boundary and internal solution. The boundary solution is displayed in Figure 4 showing that the proposed method is the most accurate one with higher convergence rate achieved. With these given boundary point densities, the solution converges as \( O(h^{2.24}), O(h^{2.04}) \) and \( O(h^{3.83}) \) for linear, quadratic and IRBFN interpolations, respectively. At \( h = 0.31 \), which corresponds to the boundary point density of \( 11 \times 4 \), error norms obtained are \( 1.27e - 2, 1.17e - 2 \)

<table>
<thead>
<tr>
<th>Boundary points</th>
<th>3 \times 4</th>
<th>4 \times 4</th>
<th>5 \times 4</th>
<th>6 \times 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear elements</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>20</td>
</tr>
<tr>
<td>Error norm of the boundary solution</td>
<td>( 3.01e-7 )</td>
<td>( 3.08e-7 )</td>
<td>( 3.72e-7 )</td>
<td>( 4.30e-7 )</td>
</tr>
<tr>
<td>Linear-BEM</td>
<td>( 7.22e-6 )</td>
<td>( 1.17e-6 )</td>
<td>( 4.33e-7 )</td>
<td>( 1.60e-7 )</td>
</tr>
<tr>
<td>IRBFN-BEM</td>
<td>( 1.86e-7 )</td>
<td>( 1.43e-7 )</td>
<td>( 1.22e-7 )</td>
<td>( 1.07e-7 )</td>
</tr>
<tr>
<td>Error norm of the internal solution</td>
<td>( 3.97e-6 )</td>
<td>( 4.07e-7 )</td>
<td>( 1.57e-7 )</td>
<td>( 5.17e-8 )</td>
</tr>
</tbody>
</table>

**Table I.**

Example 1 – error norms \( N_i \)'s of the IRBFN-BEM and linear-BEM solutions

**Note:** The selected internal points are \((2, 2), (2, 4), (3, 3), (4, 2)\) and \((4, 4)\). In the first row, \( n \times m \) means \( n \) boundary points per segment and \( m \) segments. The number of boundary elements in each case results in the same total number of boundary points.
and $2.80e - 5$ for linear, quadratic and IRBFN interpolations, respectively. The internal results are recorded in Table II showing that the IRBFN-BEM achieves a solution accuracy better than the linear/quadratic-BEM results by several orders of magnitude.

4.1.3 Example 3. The problem is to find the temperature field such that

\[ \nabla^2 u = 0 \quad \text{inside the square} \quad 0 \leq x_1 \leq \pi, 0 \leq x_2 \leq \pi, \]  

\[ u(\pi, x_2) = \sin^3(x_2) \quad \text{on the right edge} \quad (0 \leq x_2 \leq \pi), \]  

\[ u(x_1, x_2) = 0 \quad \text{on the other three sides}. \]

The analytical solution of this problem (Snider, 1999) is

\[ u(x_1, x_2) = \frac{3}{4 \sinh(\pi)} \sin(x_2) \sinh(x_1) - \frac{1}{4 \sinh(3\pi)} \sin(3x_2) \sinh(3x_1). \]

The shape of this solution is more complicated than the one in the previous example and provides a good test for the present method. The boundary point

![Figure 4. Example 2 – error norm $N_e$ of the boundary solution versus boundary point spacing $h$ obtained by the BEM with different interpolation techniques](image-url)
densities are chosen to be $9 \times 4$, $11 \times 4$, $13 \times 4$ and $15 \times 4$. The selected internal points are $(\pi/3, \pi/3)$, $(\pi/3, 2\pi/3)$, $(\pi/2, \pi/2)$, $(2\pi/3, \pi/3)$ and $(2\pi/3, 2\pi/3)$. The proposed method also performs much better than the standard BEM and similar remarks as mentioned in Example 2 apply. With $\beta = 7$, the error norms of the boundary solution and the internal solution are displayed in Figure 5 and Table III, respectively. The rates of convergence of the boundary solution are of $O(h^{2.14})$, $O(h^{1.38})$ and $O(h^{4.78})$ for linear, quadratic and IRBFN interpolations,

**Table II.** Example 2 – error norms $N_e$ of the internal solution obtained by the BEM with different interpolation techniques

<table>
<thead>
<tr>
<th>Boundary points</th>
<th>$5 \times 4$</th>
<th>$7 \times 4$</th>
<th>$9 \times 4$</th>
<th>$11 \times 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>$2.96e-2$</td>
<td>$1.25e-2$</td>
<td>$6.90e-3$</td>
<td>$4.30e-3$</td>
</tr>
<tr>
<td>Quadratic</td>
<td>$2.80e-3$</td>
<td>$5.90e-4$</td>
<td>$1.82e-4$</td>
<td>$7.66e-5$</td>
</tr>
<tr>
<td>IRBFN</td>
<td>$1.27e-5$</td>
<td>$4.79e-7$</td>
<td>$1.49e-7$</td>
<td>$3.40e-8$</td>
</tr>
</tbody>
</table>

**Note:** The IRBFN-BEM yields a solution more accurate than the linear/quadratic-BEM by several orders of magnitude.

**Figure 5.** Example 3 – error norm $N_e$ of the boundary solution versus boundary point spacing $h$ obtained from the BEM with different interpolation techniques

**Note:** With the given boundary point densities of $9 \times 4$, $11 \times 4$, $13 \times 4$ and $15 \times 4$, the rate of convergence appears to be $O(h^{2.14})$, $O(h^{1.38})$ and $O(h^{4.78})$ for linear, quadratic and IRBFN interpolations, respectively, as depicted by solid lines. The proposed method achieves accuracy and convergence rate superior to the element-based method.
respectively. At $h = 0.07$, which corresponds to the boundary point density of $15 \times 4$, the achieved error norms are $3.91e-2$, $2.79e-2$ and $6.88e-5$ for linear, quadratic and IRBFN interpolations, respectively. The accuracy of the internal solution by the present method is also better, by several orders of magnitude, than the ones by linear and quadratic BEMs. Furthermore, the CPU time requirements for the two methods are compared in Table IV. The structures of the MATLAB codes are the same and therefore it is believed that the higher efficiency achieved by the IRBFN-BEM is due to the fact that the number of segments (elements) used in the IRBFN-BEM is significantly less than that used in the standard BEM, resulting in a better vectorised computation for the former (MATLAB’s internal vectorisation).

4.2 Boundary geometry with curved and straight segments

NNs are employed to interpolate not only the variables $u$ and $q$ by using equations (21) and (22), but also the geometry of the curved segments by using equations (23) and (24). All quantities in the BIs such as $u$, $q$ and $d\Gamma$ are represented by IRBFNs necessarily in terms of the curvilinear coordinate (arclength) $s$. Special attention is given to the transformation of the quantity $d\Gamma$ from rectangular to curvilinear coordinates where the use of a Jacobian is required as follows

$$
d\Gamma = \left( \left( \frac{\partial x_1}{\partial s} \right)^2 + \left( \frac{\partial x_2}{\partial s} \right)^2 \right)^{1/2} ds,
$$

in which the derivatives of $x_1$ and $x_2$ on the segment $\Gamma_j$ can be expressed in terms of the basis function $H$ (equation (6)) as

<table>
<thead>
<tr>
<th>Boundary points</th>
<th>$9 \times 4$</th>
<th>$11 \times 4$</th>
<th>$13 \times 4$</th>
<th>$15 \times 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>$6.60e-3$</td>
<td>$4.20e-3$</td>
<td>$2.90e-3$</td>
<td>$2.20e-3$</td>
</tr>
<tr>
<td>Quadratic</td>
<td>$3.25e-4$</td>
<td>$1.74e-4$</td>
<td>$7.84e-5$</td>
<td>$4.09e-5$</td>
</tr>
<tr>
<td>IRBFN</td>
<td>$2.79e-6$</td>
<td>$1.91e-6$</td>
<td>$7.97e-7$</td>
<td>$9.64e-7$</td>
</tr>
</tbody>
</table>

Note: The IRBFN-BEM yields a solution more accurate than the linear/quadratic-BEM by several orders of magnitude.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Boundary solution</th>
<th>Total solution</th>
<th>Boundary solution</th>
<th>Total solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$9 \times 9$</td>
<td>1.98</td>
<td>4.57</td>
<td>2.07</td>
<td>2.19</td>
</tr>
<tr>
<td>$11 \times 11$</td>
<td>3.02</td>
<td>8.39</td>
<td>3.08</td>
<td>3.27</td>
</tr>
<tr>
<td>$13 \times 13$</td>
<td>4.29</td>
<td>13.88</td>
<td>4.27</td>
<td>4.63</td>
</tr>
<tr>
<td>$15 \times 15$</td>
<td>5.78</td>
<td>21.56</td>
<td>5.70</td>
<td>6.33</td>
</tr>
</tbody>
</table>

Note: The code is written in the MATLAB language (version R11.1 by The MathWorks, Inc.), which is run on a 548 MHz Pentium PC. Note that MATLAB language is interpretative.
Clearly, these derivatives can be calculated straightforwardly, once the interpolation of the function is done after solving equations (23) and (24). For more details covering the calculation of derivative functions by IRBFNs, the reader is referred to Mai-Duy and Tran-Cong (2002). Normally, the orders of IRBFN approximation for the boundary geometry and the variation of $u$ and $q$ are chosen to be the same. However, they can be different and are discussed shortly.

4.2.1 Example 4. Consider the boundary value problem governed by the Laplace equation

$$\nabla^2 u = 0$$

as shown in Figure 6. The domain of analysis is one quarter of the ellipse and the boundary conditions are

![Figure 6. Example 4 – geometry definition and training points](image)
on OA and BO and
\[ u = 0, \]
\[ \frac{\partial u}{\partial n} = - \frac{a^2 - b^2}{(a^4 x_2^2 + b^4 x_1^2)^{1/2}} x_1 x_2, \]
on AB with \( a \) and \( b \) being the half lengths of the major and minor axes, respectively. This problem with \( a = 10 \) and \( b = 5 \) was solved by quadratic BEM (Brebbia and Dominguez, 1992) using five and ten quadratic elements with two selected internal points \((2, 2)\) and \((4, 3.5)\). For the present method, the boundary is divided into three segments (two straight lines and one curve) and the training points are taken to be the same as the boundary nodes used in the case of the quadratic BEM. Thus, the densities are 5, 5 and 3 on segments OA, AB and BO, respectively, which corresponds to the case of five quadratic elements and densities 9, 9 and 5 corresponding to the case of ten quadratic elements. In order to compare the present results with the results obtained by quadratic BEM (Brebbia and Dominguez, 1992) and the exact solution, some values of the function \( u \) are extracted and the errors obtained by the two methods are displayed in Tables V and VI, which show that the present method yields better accuracy. For example, with four digit scaled fixed point, for the coarse density the range of the error is \((0.02-0.2 \text{ per cent})\) and \((0.84-2.32 \text{ per cent})\) for IRBFN-BEM and quadratic BEM, respectively, while for the fine density the error range is \((0.00-0.02 \text{ per cent})\) and \((0.02-0.14 \text{ per cent})\) for IRBFN-BEM and quadratic BEM, respectively.

4.2.2 Example 5. The distribution of the function \( u \) in an ellipse with a semi-major axis \( a = 2 \) and a semi-minor axis \( b = 1 \) is described by

\[ \nabla^2 u = -2, \quad (36) \]

subject to the condition \( u = 0 \) along the boundary \( \Gamma \). The exact solution is

\[ u(x_1, x_2) = -0.8 \left( \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1 \right). \]

\[ \begin{array}{cccccc}
 x_1 & x_2 & \text{Exact} & \text{IRBFN-BEM} & \text{Quadratic BEM} \\
 8.814 & 2.362 & -12.489 & -12.514 & 0.20 & -12.779 & 2.32 \\
 6.174 & 3.933 & -14.570 & -14.579 & 0.06 & -14.839 & 1.85 \\
 3.304 & 4.719 & -9.356 & -9.354 & 0.02 & -9.435 & 0.84 \\
 2.000 & 2.000 & -2.400 & -2.404 & 0.17 & -2.431 & 1.29 \\
 4.000 & 3.500 & -8.400 & -8.413 & 0.15 & -8.472 & 0.86 \\
\end{array} \]

Table V. Example 4 – comparison (five quadratic elements)

Note: Comparison of the error obtained by the present IRBFN-BEM (\( \beta = 7 \)) and the quadratic BEM using the same boundary nodes (five quadratic elements)
This problem is governed by the Poisson's equation and hence the BEM with PS can be applied here for obtaining the numerical solution. The solution $u$ can be decomposed into a homogeneous part $u^H$ and a PS part $u^P$ as

$$u = u^H + u^P.$$

The PS to equation (36) can be verified to be

$$u^P = -\frac{x_1^2 + x_2^2}{2}$$

while the complementary one satisfies the Laplace’s equation $\nabla^2 u^H = 0$ with the boundary condition $u^H = -u^P$ on $\Gamma$. The latter is to be solved by BEM. Partridge et al. (1992) used this approach to solve the problem in which 16 linear boundary elements are employed and the solution obtained was displayed at seven internal points. In the present method, the boundary $\Gamma$ is divided into two segments as shown in Figure 7. Four data densities, namely $9 \times 2$, $11 \times 2$, $13 \times 2$ and $15 \times 2$, and $\beta = 8$ are employed to simulate the problem. Error norms of the boundary solution obtained are $0.0105$, $0.0037$, $9 \cdot 4436 \cdot 2$ and $5 \cdot 8135 \cdot 2$ for the four densities, respectively, with the convergence rate achieved being $O(N^{-5.9289})$, where $N$ is the number of the training boundary points employed (Figure 8). In order to compare with the linear BEM (Partridge et al., 1992), the solution at seven internal points is also computed by the present method and the corresponding error norms obtained are $0.0063$, $0.0026$, $8 \cdot 0387 \cdot 5$ and $3 \cdot 4900 \cdot 5$ for the four densities, respectively. Hence with the coarse density of $9 \times 2$ that corresponds to 16 linear boundary elements, the present method achieves the error norm of $0.0063$, while the linear BEM achieves only $N_e = 0.0109$. The latter number is calculated by the present authors using the table shown in Partridge et al. (1992). Numerical result for the finest density is displayed in Table VII.

### 4.2.3 Interpolation for geometry and boundary variables

In the last two examples, the IRBFN interpolations for the geometry and the variables $u$ and $q$.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>Exact</th>
<th>IRBFN-BEM</th>
<th>Quadratic BEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.814</td>
<td>2.362</td>
<td>-12.489</td>
<td>-12.487</td>
<td>0.02</td>
</tr>
<tr>
<td>6.174</td>
<td>3.933</td>
<td>-14.570</td>
<td>-14.568</td>
<td>0.01</td>
</tr>
<tr>
<td>3.304</td>
<td>4.719</td>
<td>-9.356</td>
<td>-9.355</td>
<td>0.01</td>
</tr>
<tr>
<td>2.000</td>
<td>2.000</td>
<td>-2.400</td>
<td>-2.400</td>
<td>0.00</td>
</tr>
<tr>
<td>4.000</td>
<td>3.500</td>
<td>-8.400</td>
<td>-8.400</td>
<td>0.00</td>
</tr>
</tbody>
</table>

**Table VI.**

Example 4 – comparison (ten quadratic elements)

**Note:** Comparison of the error obtained by the present IRBFN-BEM ($\beta = 7$) and the quadratic BEM using the same boundary nodes (ten quadratic elements)
have the same order, i.e. the training points used are same for both the cases. However, the order of IRBFN interpolation can be chosen differently for the geometry and the variables $u$ and $q$ in order to obtain high quality solutions with low cost as possible. The geometry is usually known and hence the

**RBF interpolation of boundary values**

**Figure 7.**
Example 5 – geometry definition, boundary training points and internal points. The boundary is divided into two segments 
$(-a \leq x_1 \leq a, x_2 \geq 0)$
and 
$(-a \leq x_1 \leq a, x_2 \leq 0)$

**Figure 8.**
Example 5 – error norm $N_e$ of the boundary solution versus the number of boundary points $N$ by the present IRBFN-BEM. With the given boundary point densities of $9 \times 2$, $11 \times 2$, $13 \times 2$ and $15 \times 2$, the rate of convergence appears as $O(N^{-0.59289})$, where $N$ is the number of the boundary points employed.
number of training points for the geometry interpolation can be estimated. It is emphasised that the size of the final system of equations only depends on the order of IRBFN interpolation for the variables \( u \) and \( q \) and hence in the case of highly curved boundary, it is recommended that the order of IRBFN interpolation can be chosen higher for the geometry than for the variables \( u \) and \( q \). The problem in the last example is solved again with the increasing number of training points for the geometry interpolation. The density of training points employed is \( 9 \times 2 \) for the variables \( u \) and \( q \) while they are \( 12 \times 2 \) and \( 14 \times 2 \) for the geometry. The solution is improved as shown in Table VIII. For example, the error norm of the boundary solution decreases from 0.0105 for the normal case (the same order) to \( 9.5093e^{-4} \) and \( 8.2902e^{-4} \) for the increasing order of geometry interpolation.

5. Concluding remarks
In this paper, the introduction of IRBFN interpolation into the BEM scheme to represent the variables in BIEs for numerical solution of heat transfer problems is implemented and verified successfully. Numerical examples show that the proposed method considerably improves the estimate of the BIs resulting in

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>Coordinates</th>
<th>( x_2 )</th>
<th>Exact \ Graduate ( q )</th>
<th>Computed \ Graduate ( q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.997</td>
<td>0.056</td>
<td>-0.804</td>
<td>-0.802</td>
<td></td>
</tr>
<tr>
<td>1.950</td>
<td>0.223</td>
<td>-0.857</td>
<td>-0.859</td>
<td></td>
</tr>
<tr>
<td>1.802</td>
<td>0.434</td>
<td>-1.001</td>
<td>-1.000</td>
<td></td>
</tr>
<tr>
<td>1.564</td>
<td>0.623</td>
<td>-1.177</td>
<td>-1.178</td>
<td></td>
</tr>
<tr>
<td>1.247</td>
<td>0.782</td>
<td>-1.347</td>
<td>-1.347</td>
<td></td>
</tr>
<tr>
<td>0.868</td>
<td>0.901</td>
<td>-1.483</td>
<td>-1.483</td>
<td></td>
</tr>
<tr>
<td>0.445</td>
<td>0.975</td>
<td>-1.570</td>
<td>-1.570</td>
<td></td>
</tr>
<tr>
<td>0.000</td>
<td>1.000</td>
<td>-1.600</td>
<td>-1.600</td>
<td></td>
</tr>
</tbody>
</table>

**Note:** Although no symmetry condition was imposed in the numerical model, the results obtained are accurately symmetrical. Owing to symmetry, the displayed results corresponds to only a quarter of the elliptical domain

<table>
<thead>
<tr>
<th>( N_e )</th>
<th>( 9 \times 2 )</th>
<th>( 12 \times 2 )</th>
<th>( 14 \times 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary solution</td>
<td>0.0105</td>
<td>( 9.5093e^{-4} )</td>
<td>( 8.2902e^{-4} )</td>
</tr>
<tr>
<td>Internal solution</td>
<td>0.0063</td>
<td>( 1.5961e^{-4} )</td>
<td>( 9.8966e^{-5} )</td>
</tr>
</tbody>
</table>

**Note:** The densities of IRBFN interpolation are \( 9 \times 2 \) for the boundary variables and \( 9 \times 2, 12 \times 2 \) and \( 14 \times 2 \) for the geometry
better solutions not only in terms of the accuracy but also in terms of the rate of convergence. The CPV integral is written in the non-singular form where the standard Gaussian quadrature can be applied while the weakly singular integrals are evaluated by using the well-known numerical techniques as in the case of the standard BEM. The method can be extended to problems of viscous flows which will be carried out in future work.

References


