COMPACT INTEGRATED RADIAL BASIS FUNCTION MODELLING OF PARTICULATE SUSPENSIONS

A dissertation submitted by

Nha Thai-Quang


For the award of the degree of

Doctor of Philosophy

2014
Dedication

To my parents, my brother and my wife.
Certification of Dissertation

I certify that the ideas, experimental work, results and analyses, software and conclusions reported in this dissertation are entirely my own effort, except where otherwise acknowledged. I also certify that the work is original and has not been previously submitted for any other award.

Nha Thai-Quang, Candidate  Date

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Dr. Canh-Dung Tran, Co-supervisor  Date
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Last but not least, I am eternally indebted to my family and my wife Thao Duong for their unconditional love, support, and encouragement to pursue this research endeavour. I would like to dedicate this work to them with my sincerest thanks.
Notes to Readers

All content of the present thesis is recorded on the attached CD-ROM, including the following files:


2. Chapter6-Oscillating-Circular-Cylinder-Re100.wmv: An animation showing the evolution of the horizontal velocity field of the flow induced by an oscillating circular cylinder for $Re = 100$ (Chapter 6, Section 6.4.2). Online at http://www.youtube.com/watch?v=k0uQsA51YY0.

3. Chapter6-Oscillating-Circular-Cylinder-Re800.wmv: An animation showing the evolution of the horizontal velocity field of the flow induced by an oscillating circular cylinder for $Re = 800$ (Chapter 6, Section 6.4.2). Online at http://www.youtube.com/watch?v=YJlFojDuPs4.

4. Chapter6-Single-Particle-Vertical-Velocity.wmv: An animation showing the sedimentation of a particle and the evolution of the vertical velocity field of the flow in a closed box (Chapter 6, Section 6.4.3). Online at http://www.youtube.com/watch?v=OItYnm8rbvQ.

5. Chapter6-Single-Particle-Vorticity.wmv: An animation showing the sedimentation of a particle and the evolution of the vorticity of the flow in a closed box (Chapter 6, Section 6.4.3). Online at http://www.youtube.com/watch?v=X2miOpFafIU.


7. Chapter6-Two-Particles-Drafting-Kissing-Tumbling-Vorticity.wmv: An animation showing the drafting-kissing-tumbling phenomenon of two settling particles and the evolution of the vorticity of the flow in a closed box (Chapter 6, Section 6.4.4). Online at http://www.youtube.com/watch?v=Qc1FqNJsLbI.
Abstract

The present Ph.D. thesis is concerned with the development of computational procedures based on Cartesian grids, point collocation, immersed boundary method, and compact integrated radial basis functions (CIRBF), for the simulation of heat transfer and steady/unsteady viscous flows in complex geometries, and their applications for the prediction of macroscopic rheological properties of particulate suspensions.

The thesis consists of three main parts. In the first part, integrated radial basis function approximations are developed into compact local form to achieve sparse system matrices and high levels of accuracy together. These stencils are employed for the discretisation of the Navier-Stokes equation in the pressure-velocity formulation. The use of alternating direction implicit (ADI) algorithms to enhance the computational efficiency is also explored. In the second part, compact local IRBF stencils are extended for the simulation of flows in multiply-connected domains, where the direct forcing-immersed boundary (DFIB) method is adopted to handle such complex geometries efficiently. In the third part, the DFIB-CIRBF method is applied for the investigation of suspensions of rigid particles in a Newtonian liquid, and the prediction of their bulk viscosity and stresses.

The proposed computational procedures are verified successfully with several test problems in Computational Fluid Dynamics and Computational Rheology. Accurate results are achieved using relatively coarse grids.
Papers Resulting from the Research

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<tr>
<td>2D</td>
<td>Two Dimensional</td>
</tr>
<tr>
<td>ALE-FEM</td>
<td>Arbitrary Lagrangian-Eulerian Finite Element Method</td>
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<td>ADI</td>
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<td>BSQI</td>
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<td>CPU</td>
<td>Central Processing Unit</td>
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<td>DF</td>
<td>Direct Forcing</td>
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<td>DFD</td>
<td>Domain-Free Discretisation Method</td>
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<td>DKT</td>
<td>Drafting Kissing Tumbling</td>
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<td>DLM/FDM</td>
<td>Distributed Lagrange Multiplier/Fictitious Domain Method</td>
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<td>DRBFN</td>
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<td>Exponential High-order Compact Alternating Direction Implicit</td>
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<td>FBM</td>
<td>Fictitious Boundary Method</td>
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<td>FD</td>
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<td>Re</td>
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<td>RMS</td>
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<td>SIM</td>
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