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Testing the Equality of the Two Intercepts for the Parallel Regression Model

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ABSTRACT

Testing the equality of the two intercepts of two parallel regression models is considered when the slopes are suspected to be equal. For three different scenarios on the values of the slope parameters, namely (i) unknown (unspecified), (ii) known (specified), and (iii) suspected, we derive the unrestricted (UT), restricted (RT) and pretest (PTT) tests for testing the intercept parameters. The test statistics, their sampling distributions, and power functions of the tests are obtained. Comparison of power functions and sizes of the tests are provided.

Keywords and phrases: Linear regression; intercept and slope parameters; pre-test test; non-sample prior information; and power function.
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1 Introduction

Two linear regression lines are parallel if the two slopes are equal. A parallelism problem can be described as a special case of two related regression lines on the same dependent and independent variables that come from two different categories of the respondents. If the independent data sets come from two random samples (*p = 2*), researchers often wish to model the regression lines for lines groups that are parallel (i.e. the slopes of the two regression lines are equal) or whether the lines have the same intercept. To test the parallelism of the two regression equations, namely

$$y_{1j} = \theta_1 + \beta_1 x_{1j} + e_{1j} \text{ and } y_{2j} = \theta_2 + \beta_2 x_{2j} + e_{2j},\ j = 1, 2, \ldots, n_i,$$

for the two data sets: $\boldsymbol{y} = [\boldsymbol{y}_1', \boldsymbol{y}_2']'$ and $\boldsymbol{x} = [\boldsymbol{x}_1', \boldsymbol{x}_2']'$ where $\boldsymbol{y}_1 = \left[y_{11}, \ldots, y_{1n_1}\right]'$ $\boldsymbol{y}_2 = \left[y_{21}, \ldots, y_{2n_2}\right]'$, $\boldsymbol{x}_1 = \left[x_{11}, \ldots, x_{1n_1}\right]'$, $\boldsymbol{x}_2 = \left[x_{21}, \ldots, x_{2n_2}\right]'$. We use an appropriate two-sample *t* test for testing $H_0: \beta_1 = \beta_2$ (parallelism). This *t* statistic is given as

$$t = (\tilde{\beta}_1 - \tilde{\beta}_2) / S_{(\tilde{\beta}_1 - \tilde{\beta}_1)},$$

where $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are estimate of the slopes β_1 and β_2 respectively, and $S_{(\tilde{\beta}_1 - \tilde{\beta}_1)}$ is estimate of the standard error of the estimated difference between slopes (Kleinbaum, 2008, p. 223). The parallelism of the two regression equations above can be expressed as a single model of matrix form, that is,

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\Phi} + \boldsymbol{e},$$

where $\boldsymbol{\Phi} = [\theta_1, \theta_2, \beta_1, \beta_2]'$, $\boldsymbol{X} = [\boldsymbol{X}_1, \boldsymbol{X}_2]'$ with $\boldsymbol{X}_1 = [1, 0, x_1, 0]'$ and $\boldsymbol{X}_2 = [0, 1, 0, x_2]'$ and $\boldsymbol{e} = [e_1, e_2]'$. The matrix form of the intercept and slope parameters can be written, respectively, as $\boldsymbol{\theta} = [\theta_1, \theta_2]'$ and $\boldsymbol{\beta} = [\beta_1, \beta_2]'$ (cf Khan, 2006). In this model, *p* independent bivariate samples are considered such that $y_{ij} \approx N(\theta_i + \beta_i x i_j, \sigma^2)$ for $i = 1,..., p$ and $j = 1,..., n_i$. The parameters $\boldsymbol{\theta} = (\theta_1,...,\theta_p)'$ and $\boldsymbol{\beta} = (\beta_1,...,\beta_p)'$ are the intercept and slope vectors of the *p* lines. See Khan (2003, 2006, 2008) for details on parallel regression models and analyses.

To explain the importance of testing the equality of the intercepts (parallelism) when the equality of slopes is uncertain, we consider the general form of the PRM of a set of $p(p > 1)$ simple regression models as

$$\boldsymbol{Y}_i = \theta_i \mathbf{1}_{ni} + \beta_i \boldsymbol{x}_{ij} + \boldsymbol{e}_{ij},\ i\text{=1,2,...,p, and } j = 1, 2,..., n_i, \qquad (1.1)$$

where $\boldsymbol{Y}_i = (Y_{i1},...,Y_{in_i})'$ is a vector of n_i observable random variables, $\mathbf{1}_{n_i} = (1,...,1)$ is an n_i-tuple of $1'$ s, $\boldsymbol{x}_{ij} = (x_{i1},...,x_{in_i})'$ is a vector of n_i independent variables, θ_i and β_i are unknown intercept and slope, respectively, and $\boldsymbol{e}_i = (e_{i1},...,e_{in_i})'$ is the vector of errors which are mutually independent and identically distributed as normal variable, that is, $\boldsymbol{e}_i \approx N(\mathbf{0}, \sigma^2 \boldsymbol{I}_{n_i})$ where $\boldsymbol{I}_{n_i}$ is the identity matrix of order n_i. Equation (1.1) represent *p* linear models with different intercept and slope parameters. If $\beta_1 = ... = \beta_p = \beta$, then there are *p* parallel simple linear models if θ_i' s are different. Here, the parameters $\boldsymbol{\theta} = (\theta_1,...,\theta_p)'$ and $\boldsymbol{\beta} = (\beta_1,...,\beta_p)'$ are the intercept and slope vectors of the *p* lines.

Bancroft (1944) introduced the idea of pretesting NSPI to remove uncertainty. The outcome of the pretesting on the uncertain NSPI is used in the hypothesis testing to improve the performance of the statistical test (Khan and Saleh, 2001; Saleh, 2006, p. 55-58; Yunus and Khan, 2011a).

The suspected value of the slopes may be (i) unknown or unspecified if NSPI is not available, (ii) known or specified if the exact value is available from NSPI, and (iii) uncertain if the suspected value is unsure. For the three different scenarios, three different of statistical tests, namely the (i) unrestricted test (UT), (ii) restricted test (RT) and (iii) pre-test test (PTT) are defined.

In the area of estimation with NSPI there has been a lot of work, notably Bancroft (1944, 1964), Hand and Bancroft (1968), and Judge and Bock (1978) introduced a preliminary test estimation of parameters to estimate the parameters of a model with uncertain prior information. Khan (2003, 2008), Khan and Saleh (1997, 2001, 2005, 2008), Khan et al. (2002), Khan and Hoque (2003), Saleh (2006) and Yunus (2010) covered various work in the area of improved estimation using NSPI, but there is a very limited number of studies on the testing of parameters in the presence of uncertain NSPI. Although Tamura (1965), Saleh and Sen (1978, 1982), Yunus and Khan (2007, 2011a, 2011b), and Yunus (2010) used the NSPI for testing hypotheses using nonparametric methods, the problem has not been addressed in the parametric context.

The study tests the equality of the intercepts for $p \geq 2$ when the equality of slopes is suspected. We test the intercept vector $\boldsymbol{\theta} = (\theta_1,...,\theta_p)'$ when it is uncertain if the *p* slope parameters are equal (parallel). We then consider the three different scenarios of the slope parameters, and define three different tests:

for the UT, let ϕ^{UT} be the test function and T^{UT} be the test statistic for testing $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $H_a : \boldsymbol{\theta} > \boldsymbol{\theta}_0$ when $\boldsymbol{\beta} = (\beta_1,...,\beta_p)'$ is unspecified,

for the RT, let ϕ^{RT} be the test function and T^{RT} be the test statistic for testing $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $H_a : \boldsymbol{\theta} > \boldsymbol{\theta}_0$ when $\boldsymbol{\beta} = \beta_0 \mathbf{1}_p$ (fixed vector),

for the PTT, let ϕ^{PTT} be the test function and T^{PTT} be the test statistic for testing $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $H_a : \boldsymbol{\theta} > \boldsymbol{\theta}_0$ following a pre-test (PT) on the slope parameters. For the PT, let ϕ^{PT} be the test function for testing $H_{0}^{*} : \boldsymbol{\beta} = \beta_0 \mathbf{1}_p$ (a suspected constant) against $H_{a}^{*} : \boldsymbol{\beta} > \beta_0 \mathbf{1}_p$ (to remove uncertainty). If the H_0^* is rejected in the PT, then the UT is used to test the intercept, otherwise the RT is used to test H_0. Thus, the PTT depends on the PT which is a choice between the UT and RT.

The unrestricted maximum likelihood estimator or least square estimator of intercept and slope vectors, $\boldsymbol{\theta} = (\theta_1, ..., \theta_p)'$ and $\boldsymbol{\beta} = (\beta_1, ..., \beta_p)'$, are given as

$$\tilde{\boldsymbol{\theta}} = \overline{\boldsymbol{Y}} - \boldsymbol{T}\tilde{\boldsymbol{\beta}}^{UT} \text{ and } \tilde{\boldsymbol{\beta}} = \frac{(\boldsymbol{x}_i'\boldsymbol{y}_i) - \left(\frac{1}{n_i}\right)\left[\mathbf{1}_i'\boldsymbol{x}_i\mathbf{1}_i'\boldsymbol{y}_i\right]}{n_iQ_i}, \qquad (1.2)$$

where $\tilde{\boldsymbol{\theta}} = (\tilde{\theta}_1, ..., \tilde{\theta}_p)'$, $\tilde{\beta} = (\tilde{\beta}_1, ..., \tilde{\beta}_p)'$, $\boldsymbol{T} = \text{Diag}(\overline{x}_1, ..., \overline{x}_p)$, $n_iQ_i = \boldsymbol{x}_i'\boldsymbol{x}_i - \left(\frac{1}{n_i}\right)\left[\mathbf{1}_i'\boldsymbol{x}_i\right]$, and $\tilde{\theta}_i = \overline{Y}_i - \tilde{\beta}_i\overline{x}_i$ for $i = 1, .., p$.

Furthermore, the likelihood ratio (LR) test statistics for testing $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $H_a : \boldsymbol{\theta} > \boldsymbol{\theta}_0$ is given by

$$F = \frac{\tilde{\boldsymbol{\theta}}'\boldsymbol{H}'\boldsymbol{D}_{22}^{-1}\boldsymbol{H}\tilde{\boldsymbol{\theta}}}{(p-1)s_e^2}, \qquad (1.3)$$

where $\boldsymbol{H} = \boldsymbol{I}_p - \frac{1}{nQ}\mathbf{1}_p\mathbf{1}_p'\boldsymbol{D}_{22}^{-1}$, $\boldsymbol{D}_{22}^{-1} = \text{Diag}(n_1Q_1, .., n_pQ_p)$, $nQ = \sum_{i=1}^{p} n_iQ_i$, $n_iQ_i = \boldsymbol{x}_i'\boldsymbol{x}_i - \frac{1}{n_i}(\mathbf{1}_i'\boldsymbol{x}_i)^2$ and $S_e^2 = (n-2p)^{-1}\sum_{i=1}^{p}(\boldsymbol{Y} - \tilde{\theta}_i\mathbf{1}_{n_i} - \tilde{\beta}\boldsymbol{x}_i)'(\boldsymbol{Y} - \tilde{\theta}_i\mathbf{1}_{n_i} - \tilde{\beta}\boldsymbol{x}_i)$ (Saleh, 2006, p. 14-15). Under H_0, F follows a central F distribution with $(p-1, n-2p)$ degrees of freedom (d.f.), and under H_a it follows a noncentral F distribution with $(p-1, n-2p)$ degrees of freedom and noncentrality parameter $\boldsymbol{\Delta}^2/2$, where

$$\Delta^2 = \frac{(\boldsymbol{\theta} - \boldsymbol{\theta}_0)'\boldsymbol{D}_{22}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)}{\sigma^2} \qquad (1.4)$$

and $\boldsymbol{D}_{22} = \boldsymbol{H}'\boldsymbol{D}_{22}^{-1}\boldsymbol{H}$. When the slope $(\boldsymbol{\beta})$ is equal to $\beta_0\mathbf{1}_p$ (specified), the restricted mle of intercept and slope vectors are given as

$$\hat{\theta}_i = \tilde{\theta}_i + \boldsymbol{T}\boldsymbol{H}\tilde{\beta}_i \text{ and } \hat{\beta}_i = \frac{\mathbf{1}_k\mathbf{1}_k'\boldsymbol{D}_{22}^{-1}\tilde{\beta}_i}{n\boldsymbol{Q}} \qquad (1.5)$$

The following section provides the proposed tests. Section 3 derives the distribution of the test statistics. The power function of the tests are obtained in Section 4. An illustrative example is given in Section 5. The comparison of the power of the tests and concluding remarks are provided in Sections 6 and 7.

2 The Three Tests

To test the equality of the intercepts when the equality of slopes is suspected, we consider three different scenarios of the slopes. The test statistics of the UT, RT and PTT are then defined as follows.

For $\boldsymbol{\beta}$ unspecified, the test statistic of the UT is given by

$$T^{UT} = \frac{\tilde{\boldsymbol{\theta}}'\boldsymbol{H}'\boldsymbol{D}_{22}^{-1}\boldsymbol{H}\tilde{\boldsymbol{\theta}}}{(p-1)s_e^2}, \qquad (2.1)$$

where $s_e^2 = (n-2p)^{-1}\sum_{i=1}^{n}(\boldsymbol{Y} - \tilde{\theta}_i\mathbf{1}_{n_i} - \tilde{\beta}\boldsymbol{x}_i)'(\boldsymbol{Y} - \tilde{\theta}_i\mathbf{1}_{n_i} - \tilde{\beta}\boldsymbol{x}_i)$.

The T^{UT} follows a central *F* distribution with $(p-1, n-2p)$ degrees of freedom. Under H_a, it follows a noncentral *F* distribution with $(p-1, n-2p)$ degrees of freedom and noncentrality parameter $\Delta^2/2.$ Under normal model we have

$$\begin{pmatrix} \tilde{\theta}-\boldsymbol{\theta} \\ \tilde{\beta}-\boldsymbol{\beta} \end{pmatrix} \approx N_{2p}\left[\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \ \sigma^2 \begin{pmatrix} \boldsymbol{D}_{11} & -\boldsymbol{TD}_{22} \\ -\boldsymbol{TD}_{22} & \boldsymbol{D}_{22} \end{pmatrix}\right], \quad (2.2)$$

where $\boldsymbol{D}_{11} = \boldsymbol{N}^{-1} + \boldsymbol{TD}_{22}\boldsymbol{T}\boldsymbol{\beta}$ and $\boldsymbol{N} = \text{Diag}(n_1, ..., n_p).$

When the slope is specified to be $\boldsymbol{\beta} = \beta_0 \mathbf{1}_p$ (fixed vector), the test statistic of the RT is given by

$$T^{RT} = \frac{(\hat{\theta}'\boldsymbol{H}'\boldsymbol{D}_{22}^{-1}\boldsymbol{H}\hat{\theta}) + (\tilde{\beta}'\boldsymbol{H}'\boldsymbol{D}_{22}^{-1}\boldsymbol{H}\tilde{\beta})}{(p-1)s_e^2}, \quad \text{where } (2.3)$$

$$s_r^2 = (n-p)^{-1}\sum_{i=1}^{p}(\boldsymbol{Y} - \hat{\theta}_i \mathbf{1}_{n_i} - \hat{\beta}\boldsymbol{x}_i)'(\boldsymbol{Y} - \hat{\theta}_i \mathbf{1}_{n_i} - \hat{\beta}\boldsymbol{x}_i) \text{ and } \hat{\beta} = \beta_0 \mathbf{1}_p$$

The T^{RT} follows a central *F* distribution with $(p-1, n-2p)$ degrees of freedom. Under H_a, it follows a noncentral *F* distribution with $(p-1, n-2p)$ degrees of freedom and noncentrality parameter $\Delta^2/2.$ Again, note that

$$\begin{pmatrix} \hat{\theta}-\boldsymbol{\theta} \\ \hat{\beta}-\boldsymbol{\beta} \end{pmatrix} \approx N_{2p}\left[\begin{pmatrix} \boldsymbol{TH\beta} \\ \mathbf{0} \end{pmatrix}, \ \sigma^2 \begin{pmatrix} \boldsymbol{D}_{\underset{11}{*}} & \boldsymbol{D}_{\underset{12}{*}} \\ \boldsymbol{D}_{\underset{12}{*}} & \boldsymbol{D}_{22} \end{pmatrix}\right], \quad (2.4)$$

where $\boldsymbol{D}_{\underset{11}{*}} = \boldsymbol{N}^{-1} + \dfrac{\boldsymbol{T1}_p\boldsymbol{1}_p'\boldsymbol{T\beta}}{nQ}$ and $\boldsymbol{D}_{\underset{12}{*}} = \frac{1}{nQ}\boldsymbol{1}_p\boldsymbol{1}_p'\boldsymbol{T}.$

When the value of the slope is suspected to be $\boldsymbol{\beta} = \beta_0 \mathbf{1}_p$ but unsure, a pre-test on the slope is required before testing the intercept. For the preliminary test (PT) of $H_{\underset{0}{*}}: \boldsymbol{\beta} = \beta_0 \mathbf{1}_p$ against $H_{\underset{a}{*}}: \boldsymbol{\beta} > \beta_0 \mathbf{1}_p$, the test statistic under the null hypothesis is defined as

$$T^{PT} = \frac{\tilde{\beta}'\boldsymbol{H}'\boldsymbol{D}_{22}^{-1}\boldsymbol{H}\tilde{\beta}}{(p-1)s_e^2}, \quad (2.5)$$

which follows a central *F* distribution with $(p-1, n-2p)$ degrees of freedom. Under H_a, it follows a noncentral *F* distribution with $(p-1, n-2p)$ degrees of freedom and noncentrality parameter $\Delta^2/2.$ Again, note that

$$\begin{pmatrix} \tilde{\theta}-\beta_0\mathbf{1}_p \\ \tilde{\beta}-\hat{\beta} \end{pmatrix} \approx N_{2p}\left[\begin{pmatrix} (\tilde{\beta}^*-\beta_0)\mathbf{1}_p \\ \boldsymbol{H\beta} \end{pmatrix}, \ \sigma^2 \begin{pmatrix} \mathbf{1}_p\mathbf{1}_p'/nQ & \mathbf{0} \\ \mathbf{0} & \boldsymbol{HD}_{22} \end{pmatrix}\right], \quad (2.6)$$

where $\tilde{\beta}^*\mathbf{1}_p = \dfrac{\mathbf{1}_p\mathbf{1}_p'\boldsymbol{D}_{22}^{-1}\boldsymbol{\beta}}{nQ}$ (Saleh, 2006, p. 273).

Let us choose a positive number $\alpha_j (0 < \alpha_j < 1, \text{ for } j = 1,2,3)$ and real value $F_{v1,v2,v3} (v_1 \text{ be numerator d.f. and } v_2 \text{ be denominator d.f.})$ such that

$$P(T^{UT} > F_{p-1,n-2p,\alpha 1} \mid \boldsymbol{\theta} = \boldsymbol{\theta}_0) = \alpha_1, \quad (2.7)$$

$$P(T^{RT} > F_{p-1,n-2p,\alpha 2} \mid \boldsymbol{\theta} = \boldsymbol{\theta}_0) = \alpha_2, \qquad (2.8)$$

$$P(T^{PT} > F_{p-1,n-2p,\alpha 3} \mid \boldsymbol{\beta} = \beta_0 \mathbf{1}_p) = \alpha_3. \quad (2.9)$$

Now the test function for testing $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $H_a : \boldsymbol{\theta} > \boldsymbol{\theta}_0$ is defined by

$$\Phi \begin{cases} 1, \text{ if } (T^{PT} \le F_c,\ T^{RT} > F_b) \text{ or } (T^{PT} > F_c,\ T^{UT} > F_a); \\ 0, \text{ otherwise,} \end{cases} \qquad (2.10)$$

where $F_a = F_{\alpha 1, p-1, n-2p,}$ $F_b = F_{\alpha 2, p-1, n-2p}$ and $F_c = F_{\alpha 3, p-1, n-2p.}$

3 Distribution of Test Statistics

To derive the power function of the UT, RT and PTT, the sampling distribution of the test statistics proposed in Section 2 are required. For the power function of the PTT the joint distribution of (T^{UT}, T^{PT}) and (T^{RT}, T^{PT}) is essential. Let $\{N_n\}$ be a sequence of alternative hypotheses defined as

$$N_n : (\boldsymbol{\theta} - \boldsymbol{\theta}_0, \boldsymbol{\beta} - \beta_0 \mathbf{1}_p) = \left(\frac{\boldsymbol{\lambda}_1}{\sqrt{n}}, \frac{\boldsymbol{\lambda}_2}{\sqrt{n}} \right) = \boldsymbol{\lambda}, \qquad (3.1)$$

where $\boldsymbol{\lambda}$ is a vector of fixed real numbers and $\boldsymbol{\theta}$ is the true value of the intercept. Under N_n the value of $(\boldsymbol{\theta} - \boldsymbol{\theta}_0)$ is greater than zero and under H_0 the value of $(\boldsymbol{\theta} - \boldsymbol{\theta}_0)$ is equal zero.

Following Yunus and Khan (2011b) and equation (2.1), we define the test statistic of the UT when $\boldsymbol{\beta}$ is unspecified, under N_n, as

$$T_1^{UT} = T^{UT} - n \left\{ \frac{(\boldsymbol{\theta} - \boldsymbol{\theta}_0)' \boldsymbol{H}' \boldsymbol{D}_{22}^{-1} \boldsymbol{H} (\boldsymbol{\theta} - \boldsymbol{\theta}_0)}{(p-1) s_e^2} \right\}. \quad (3.2)$$

The T_1^{UT} follows a noncentral *F* distribution with noncentrality parameter which is a function of $(\boldsymbol{\theta} - \boldsymbol{\theta}_0)$ and $(p-1, n-2p)$ degrees of freedom, under N_n.

From equation (2.3) under N_n, $(\boldsymbol{\theta} - \boldsymbol{\theta}_0) > 0$ and $(\boldsymbol{\beta} - \beta_0 \mathbf{1}_{\mathrm{p}}) > 0$, the test statistic of the RT becomes

$$T_2^{RT} = T^{RT} - n \left\{ \frac{(\boldsymbol{\theta} - \boldsymbol{\theta}_0)' \boldsymbol{H}' \boldsymbol{D}_{22}^{-1} \boldsymbol{H} (\boldsymbol{\theta} - \boldsymbol{\theta}_0) + (\boldsymbol{\beta} - \beta_0 \mathbf{1}_{\mathrm{p}})' \boldsymbol{H}' \boldsymbol{D}_{22}^{-1} \boldsymbol{H} (\boldsymbol{\beta} - \beta_0 \mathbf{1}_{\mathrm{p}})}{(p-1) s_r^2} \right\}$$

(3.3)

The T_2^{RT} also follows a noncentral *F* distribution with a noncentrality parameter which is a function of $(\boldsymbol{\theta} - \boldsymbol{\theta}_0)$ and $(p-1, n-2p)$ degrees of freedom, under N_n. Similarly, from the equation (2.5) the test statistic of the PT is given by

$$T_3^{PT} = T^{PT} - n \left\{ \frac{(\boldsymbol{\beta} - \beta_0 \mathbf{1}_{\mathrm{p}})' \boldsymbol{H}' \boldsymbol{D}_{22}^{-1} \boldsymbol{H} (\boldsymbol{\beta} - \beta_0 \mathbf{1}_{\mathrm{p}})}{(p-1) s_e^2} \right\} \qquad (3.4)$$

Under H_a, the T_3^{PT} follows a noncentral *F* distribution with a noncentrality parameter which is a function of $(\boldsymbol{\beta} - \beta_0 \mathbf{1}_{\mathrm{p}})$ and $(p-1, n-2p)$ d.f.

From equations (2.1), (2.3) and (2.5) the T^{UT} and T^{PT} are correlated, and the T^{RT} and T^{PT} are uncorrelated. The joint distribution of the T^{UT} and T^{PT}, that is,

$$\left(T^{UT}, T^{PT} \right)' \qquad (3.5)$$

is a correlated bivariate *F* distribution with $(p-1, n-2p)$ degrees of freedom. The probability density function (pdf) and cumulative distribution function (cdf) of the correlated bivariate *F* distribution is found in Krishnaiah (1964), Amos and Bulgren (1972) and El-Bassiouny and Jones (2009). Later, Johnson et al. (1995, p. 325) described a relationship of the bivariate *F* distribution with the bivariate beta distribution. This is due to the pdf of the bivariate *F* distribution has a similar form with the pdf of *beta distribution of the second kind*.

Following El-Bassiouny and Jones (2009), the covariance and correlation between the T^{UT} and T^{PT} are then given as

$$Cov(T_1^{UT}, T_3^{PT}) = \frac{2 f_1 f_2}{(f_1-2)(f_2-2)(f_2-4)}$$

$$= \frac{2(n^2-4np+4p^2)}{(n-2p-2)^2(n-2p-4)}, \text{ and} \quad (3.6)$$

$$\rho^2_{T_1^{UT}T_3^{PT}} = \frac{d_1 d_2 (f_1-4)}{(f_1+d_1-2)(f_2+d_2-2)(f_2-4)}$$

$$= \frac{(n^2-2np+p^2)(n-2p-4)}{(2n-3p-2)^2(n-2p-4)}. \quad (3.7)$$

Note in the above expressions $d_1 = d_2 = p-1$ and $f_1 = f_2 = n-2p$ are the appropriate degrees of freedom for the T^{UT} and T^{PT} respectively.

4 The Power and Size of Tests

The power function of the UT, RT and PTT are derived below. From equation (2.1) and (3.2), (2.3) and (3.3), and (2.5) and (3.4), the power function of the UT, RT and PTT are given, respectively, as:

the power of the UT

$$\pi^{UT}(\boldsymbol{\lambda}) = P(T^{UT} > F_{\alpha_1, p-1, n-2p} \mid N_n)$$

$$= 1 - P(T_1^{UT} \le F_{\alpha_1, p-1, n-2p} - k_1\delta_1), \quad (4.1)$$

where $\delta_1 = \boldsymbol{\lambda}_1' \boldsymbol{D}_{22} \boldsymbol{\lambda}_1$ and $k_1 = \frac{1}{(p-1)s_e^2}$.

the power of the RT

$$\pi^{RT}(\boldsymbol{\lambda}) = P(T^{RT} > F_{\alpha_1, n-1, n-2p} \mid N_n)$$

$$= 1 - P\left(T_2^{RT} \le F_{\alpha_2, p-1, n-2p} - \frac{(\boldsymbol{\lambda}_1' \boldsymbol{H}' \boldsymbol{D}_{22}^{-1} \boldsymbol{H} \boldsymbol{\lambda}_1) + (\boldsymbol{\lambda}_2' \boldsymbol{H}' \boldsymbol{D}_{22}^{-1} \boldsymbol{H} \boldsymbol{\lambda}_2)}{(p-1)s_r^2} \right)$$

$$= 1 - P\left(T_1^{RT} \le F_{\alpha_1, p-1, n-2p} - k_2(\delta_1 + \delta_2)\right), \quad (4.2)$$

where $\delta_2 = \boldsymbol{\lambda}_2' \boldsymbol{D}_{22} \boldsymbol{\lambda}_2$ and $k_2 = \frac{1}{(p-1)s_r^2}$.

The power function of the PT is

$$\pi^{PT}(\boldsymbol{\lambda}) = P(T^{PT} > F_{\alpha_3, p-1, n-2p} \mid K_n)$$

$$= 1 - P\left(T_3^{PT} \le F_{\alpha_3, p-1, n-2p} - \frac{\boldsymbol{\lambda}_2' \boldsymbol{H}' \boldsymbol{D}_{22}^{-1} \boldsymbol{H} \boldsymbol{\lambda}_2}{(p-1)s_e^2} \right)$$

$$= 1 - P(T_3^{PT} \le F_{\alpha_3, p-1, n-2p} - k_1 \delta_2). \quad (4.3)$$

the power of the PTT

$$\pi^{PTT}(\boldsymbol{\lambda}) = P(T^{PT} < F_{\alpha_3, p-1, n-2p,} T^{RT} > F_{\alpha_2, p-1, n-2p})$$

$$+ P\left(T^{PT} \ge F_{\alpha_3, p-1, n-2p,} T^{UT} > F_{\alpha_1, p-1, n-2p}\right)$$

$$= (1-\pi^{PT})\pi^{RT} + d_{1r}(a,b), \quad (4.4)$$

where $d_{1r}(a,b)$ is bivariate F probability integrals, and it is defined as

$$d_{1r}(a,b) = \int_a^{\infty} \int_b^{\infty} f(F^{PT}, F^{UT}) dF^{PT} dF^{UT}$$

$$= 1 - \int_0^a \int_0^b f(F^{PT}, F^{UT}) dF^{PT} dF^{UT}, \quad (4.5)$$

in which

$$a = F_{\alpha_3, p-1, n-2p} - \frac{\boldsymbol{\lambda}_2' \boldsymbol{H}' \boldsymbol{D}_{22}^{-1} \boldsymbol{H} \boldsymbol{\lambda}_2}{(p-1)s_e^2} = F_{\alpha_3, p-1, n-2p} - k_1 \delta_2 \text{ , and}$$

$$b = F_{\alpha_1, p-1, n-2p} - \frac{(\boldsymbol{\theta} - \boldsymbol{\theta}_0)' \boldsymbol{H}' \boldsymbol{D}_{22}^{-1} \boldsymbol{H} (\boldsymbol{\theta} - \boldsymbol{\theta}_0)}{(p-1)s_e^2} = F_{\alpha_1, p-1, n-2p} - k_1 \delta_1.$$

The $\int_0^a \int_0^b f(F^{PT}, F^{UT}) dF^{PT} dF^{UT}$ in equation (4.5) is the cdf of the correlated bivariate noncentral F (BNCF) distribution of the UT and PT.

From equation (4.4), it is clear that the cdf of the BNCF distribution involved in the expression of the power function of the PTT. Using equation (4.7), we use it in the calculation of the power function of the PTT. R codes are written, and the R package is used for computations of the power and size and graphical analysis.

Furthermore, the size of the UT, RT and PTT are given respectively as:

the size of the UT $\alpha^{UT} = P(T^{UT} > F_{\alpha_1, p-1, n-2p} \mid H_0 : \boldsymbol{\theta} - \boldsymbol{\theta}_0)$

$$= 1 - P(T_1^{UT} \le F_{\alpha_1, p-1, n-2p}), \quad (4.8)$$

the size of the RT $\alpha^{RT} = P(T^{RT} > F_{\alpha_2, p-1, n-2p} \mid H_0 : \boldsymbol{\theta} - \boldsymbol{\theta}_0)$

$$= 1 - P(T_2^{RT} \le F_{\alpha_2, p-1, n-2p} - k_2 \delta_2), \quad (4.9)$$

The size of the PT is given by $\alpha^{PT}(\boldsymbol{\lambda}) = P(T^{PT} > F_{\alpha_3, p-1, n-2p} \mid H_0)$

$$= 1 - P(T_3^{PT} \le F_{\alpha_3, p-1, n-2p}). \quad (4.10)$$

the size of the PTT

$$\alpha^{PTT} = P(T^{PT} \le a|_{H_0}, T^{RT} > d|_{H_0}) + P(T^{PT} > a, T^{UT} > h|_{H_0})$$

$$= (1 - P(T^{PT} > F_{\alpha_3, p-1, n-2p})) P(T^{RT} > F_{\alpha_2, p-1, n-p}) + d_{1r}(a,h), \quad (4.11)$$

where $h = F_{\alpha_1, p-1, n-2p}$.

5 Power Comparison by Simulation

To compare the tests graphically we conducted simulations using the R package. For $p=3$, each of three independent variables $(x_{ij},\ i=1,2,3,\ j=1,...,n_i)$ are generated from the uniform distribution between 0 and 1. The errors $(e_i,\ i=1,2,3)$ are generated from the normal distribution with $\mu=0$ and $\sigma^2=1$. In each case $n_i=n=100$ random variates were generated. The dependent variable (y_{ij}) is determined by $y_{1j}=\theta_{01}+\beta_{11}x_{1j}+e_1$ for $\theta_{01}=3$ and $\beta_{11}=2$. Similarly, define $y_{2j}=\theta_{02}+\beta_{12}x_{2j}+e_2$ for $\theta_{02}=3.6$ and $\beta_{12}=2$; $y_{3j}=\theta_{03}+\beta_{13}x_{3j}+e_3$, for $\theta_{03}=4$ and $\beta_{13}=2$, respectively. For the computation of the power function of the tests (UT, RT and PTT) we set $\alpha_1=\alpha_2=\alpha_3=\alpha=0.05$. The graphs for the power function of the three tests are produced using the formulas in equations (4.1), (4.2) and (4.4). The graphs for the size of the three tests are produced using the formulas in equations (4.8), (4.9) and (4.11). The graphs of the power and size of the tests are presented in the Figures 1 and 2.

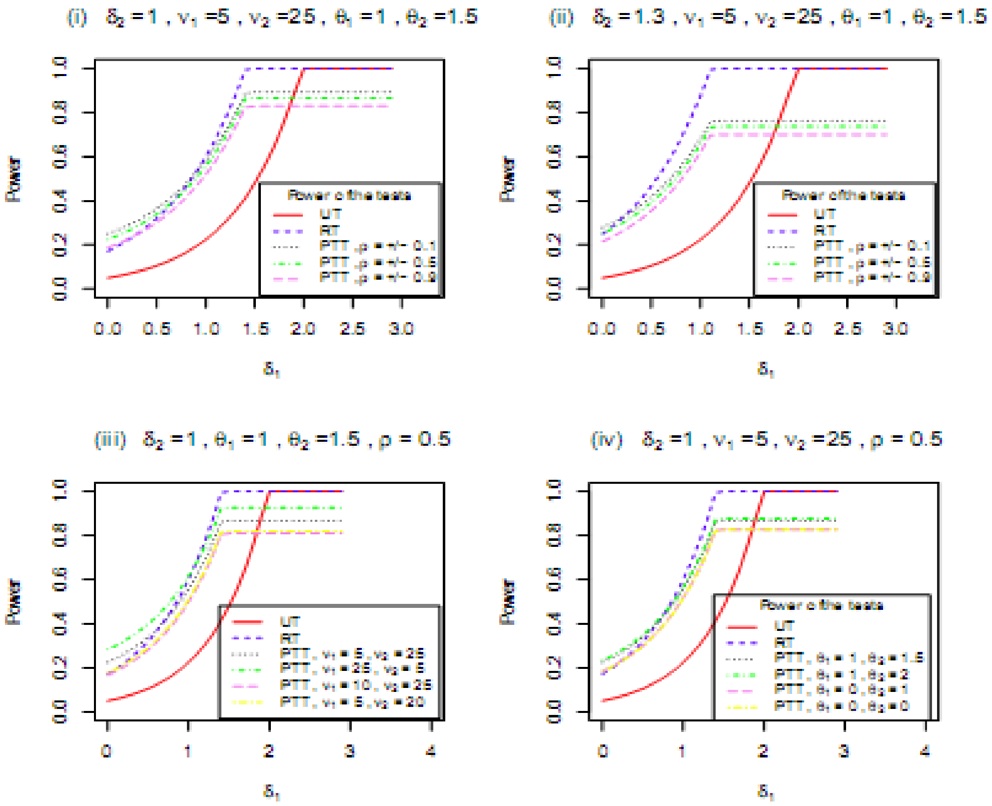

Figure 1: The power function of the UT, RT and PTT against δ_1 for some selected ρ, degrees of freedom and noncentrality parameters.

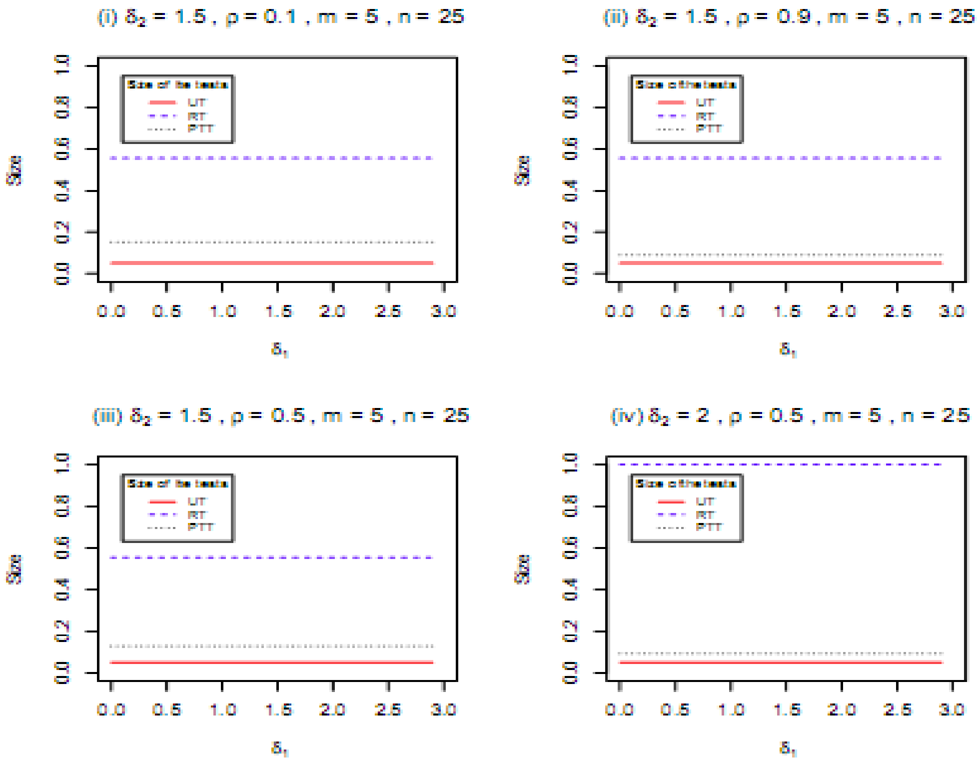

Figure 2: The size of the UT, RT and PTT against δ_1 for some selected ρ and δ_2.

6 Comparison and Conclusion

The form of the power curve of the UT in Figure 1 is concave, starting from a very small value of near zero (when δ_1 is also near 0), it approaches 1 as δ_1 grows larger. The power of the UT increases rapidly as the value of δ_1 becomes larger. The shape of the power curve of the RT is also concave for all values of δ_1 and δ_2. The power of the RT increases as the values of δ_1 and/or δ_2 increase (see Figures 1(i) and 1(ii), and equation (4.2)).

The power of the PTT (see Figure 1) increases as the values of δ_1 increase. Moreover, the power of the PTT is always larger than that of the UT and RT for the values of δ_1 around 0.7 to 1.5.

The size of the UT does not depend on δ_2. It is a constant and remains unchanged for all values of δ_1 and δ_2. The size of the RT increases as the value of δ_2 increases . Moreover, the size of the RT is always larger than that of the UT, but not for PTT for the smaller values of the δ_1 (not far from 0).

The size of the PTT is closer to that of the UT for larger values of $\delta_2 = 2$. The difference (or gap) between the size of the RT and PTT increases significantly as the value of δ_2 and ρ increases. The size of the UT is $\alpha^{UT} = 0.05$ for all values of δ_1 and δ_2. For all values of δ_1 and δ_2, the size of the RT is larger than that of the UT, $\alpha^{RT} > \alpha^{UT}$. For all the values of ρ, $\alpha^{PTT} \le \alpha^{RT}$.

Based on the above analyses, the power of the RT is always higher than that of the UT for all values of δ_1 and δ_2. Also, the power of the PTT is always larger than that of the UT for all values δ_1 (see the curves for interval values of $0.7 < \delta_1 < 1.5$), δ_2 and ρ. The size of the UT is smaller than that of the RT and PTT for all δ_1. The power of the PTT is higher than that of the UT and tends to be lower than that of the RT. The size of the PTT is less than that of the RT but higher than that of the UT.

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