Finite-difference Approach for a 6th-order Nonlinear Phase Equation with Self-excitation

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A range of physical systems, particularly of chemical nature involving reactions, perform self-excited oscillations coupled by diffusion. The role of diffusion is not trivial so that initial differences in the phase of the oscillations between different points in space do not necessarily disappear as time goes; they may self-sustain. The dynamics of the phase depend on the values of the controlling parameters of the system. We consider a 6th-order nonlinear partial differential equation resulting in such dynamics. The equation is solved using central finite-difference discretization in space. The resulting system of ordinary differential equations is integrated in time using a Matlab solver. The numerical code is tested using forced versions of the equation, which admit exact analytical solutions. The comparison of the exact and numerical solutions demonstrates satisfactory agreement.

KEYWORDS: nonlinear partial differential equation, finite difference, self-excitation

1. Introduction

We analyse the equation describing the phase of oscillations in a class of dissipative physical systems with self-excitation. Intuitively, diffusion should smooth out any differences of the phase in space. However, as is well-established [1,2], under certain conditions the combined effect of the diffusion and self-excitation (caused by reactions) may produce more complicated dynamics, where this difference never disappears. Strunin derived [3,4] a form of such an equation incorporating *nonlinear* self-excitation,

$$\partial_t u = -g\nabla^2 u(\nabla u)^2 + b(\nabla u)^4 + c\nabla^6 u\,, \tag{1}$$

where $g > 0$ and $c > 0$. All quantities in this paper are non-dimensional. The former condition ensures that the term $-a\nabla^2 u(\nabla u)^2$ acts as a nonlinear self-excitation (anti-diffusion), and the latter condition ensures dissipative effect of the term $c\nabla^6 u$. The equation will be considered in the square domain $0 < x, y < L$. On the boundaries we stipulate zero value of the first, second and third derivatives normal to the boundary.

2. The forced equation

The exact physical meaning of the phase u is the departure from the uniformly increasing phase of the oscillation $c_2 t$ (we use the notations from [4]); thus the oscillators behave as

$$a\sin\varphi \quad \text{with} \quad \varphi = c_2 t + u\,,$$

where a is the amplitude of the oscillations. In [2,4] c_2 is constant in space and time.

It is interesting to explore the case when the rate c_2 varies in space; this assumption would address the situation when the kinetics of the reactions are different at different points in space. As pointed

out in [2, 4] the phase equations in question may be relevant to certain real-life systems such as cellular slime molds and the Belousov–Zhabotinsky reaction dispersed in water-in-oil aerosol OT microemulsion. A real physical system of any kind is never uniform, in particular, the parameter c_2 may vary (naturally fluctuate) in space. In the ideal case of a constant rate c_2 the phase satisfies the equation (35) from [4],

$$\partial_t \varphi = c_2 a^2 + \cdots = c_2(1 + \varepsilon_1^2 a_2 + \varepsilon_1^4 a_4 + \varepsilon_1^6 a_6 + \dots)^2 + \dots , \tag{2}$$

where the amplitude a is represented as a series in small parameter ε_1 and the second set of dots denotes the terms which do not contain c_2. The role of ε_1 in the theory is to guarantee slow variations of the phase and amplitude. Now assume that c_2 is not constant is space (but still constant in time),

$$c_2 = c_2^{(0)} + \varepsilon_2 c_2^{(1)}(x, y) \tag{3}$$

with $c_2^{(0)}$ being a constant and ε_2 being a new small parameter. As the evolution in time is slow, we scale the time by $\partial_t = \varepsilon_1^2 \partial_{t_1}$ and look for the solution of (2) in the form

$$\varphi(x, y, t) = c_2^{(0)} t + u(x, y, t) .$$

Equation (2) becomes

$$\begin{aligned} c_2^{(0)} + \varepsilon_1^2 \partial_{t_1} u &= \left[c_2^{(0)} + \varepsilon_2 c_2^{(1)}\right](1 + \varepsilon_1^2 a_2 + \dots)^2 + \cdots = \left[c_2^{(0)} + \varepsilon_2 c_2^{(1)}\right](1 + 2\varepsilon_1^2 a_2 + \dots) + \dots \\ &= c_2^{(0)} + \varepsilon_2 c_2^{(1)} + \left\{\left[c_2^{(0)} + \varepsilon_2 c_2^{(1)}\right] 2\varepsilon_1^2 a_2 + \dots\right\} + \dots . \end{aligned} \tag{4}$$

The amplitude components a_n are expressed in terms of $\nabla\varphi = \nabla u$ from the separate amplitude equation which we omit. Substituting those into (4), cancelling $c_2^{(0)}$ in the both sides and performing necessary manipulations eventually leads to

$$\begin{aligned} \varepsilon_1^2 \partial_{t_1} u = \varepsilon_2 c_2^{(1)} - \varepsilon_1^4 \nabla_1^2 u (\nabla_1 u)^2 \left\{\left[c_2^{(0)} + \varepsilon_2 c_2^{(1)}\right] 2\beta_3 + \dots\right\} \\ + \varepsilon_1^4 (\nabla u)^4 \left\{\left[c_2^{(0)} + \varepsilon_2 c_2^{(1)}\right] 2\beta_4 + \dots\right\} + \varepsilon_1^6 \nabla^6 u \left\{\left[c_2^{(0)} + \varepsilon_2 c_2^{(1)}\right] \delta_1 \beta + \dots\right\} + \dots , \end{aligned} \tag{5}$$

where the coordinates are scaled by $\nabla = \varepsilon_1 \nabla_1$, and δ_1, β_n and β are the parameters made of the original parameters of the physical system. Note that the lower-order terms in ∇_1, carried inside a_2 and a_4, vanished because the resulting factors in front of them have been made zero. This is done intentionally by an appropriate selection of the original parameters. The analysis which takes into account the magnitude of u shows (we refer to [4] for details) that the magnitude of the small parameter ε_2 must be $\varepsilon_2 = \varepsilon_1^{20/3}$ since all the terms in (5) need to be of the same order of magnitude. Finally, neglecting $\varepsilon_2 c_2^{(1)}$ inside the square brackets in comparison to $c_2^{(0)}$ and returning to the unscaled operators $\nabla = \varepsilon_1 \nabla_1$ and $\partial_t = \varepsilon_1^2 \partial_{t_1}$ we arrive at the forced equation

$$\partial_t u = -g \nabla^2 u (\nabla u)^2 + b(\nabla u)^4 + c \nabla^6 u + f(x, y) , \tag{6}$$

where the force term, $f(x, y)$, is in fact the scaled rate $\varepsilon_2 c_2^{(1)}(x, y)$ from (5).

Our interest in the forced equation (6) is three-fold: (1) explore the effects of slow variations of the reacting media in space, (2) construct exact solutions of the forced equation and numerically investigate their stability as shown in the next section, (3) verify the numerical code by comparing the numerical solutions with the exact ones.

3. Solution of the forced equation

We developed a numerical code in Matlab which solves equations (1) and (6). The spatial part of the equation is discretized using central finite differences and the resulting system of ordinary differential equations is integrated in time by the DAE2 solver [5]. The solver ensures a good accuracy, nevertheless we carried out our own test as described below. We chose the force function in the form such that the forced equation has a desired exact solution. A-priori there is no guarantee that the desired solution is stable. As an example, let us wish that an equation

$$\dot{g}(t) = g(t) + f_1(t) \tag{7}$$

has an exact stationary solution $g_0 = 1$. Obviously we must choose $f_1 = \dot{g}_0 - g_0 = -1$. It is easy to see that the solution $g_0 = 1$ is unstable. But, if it turns out, in opposition to this example, that a desired exact solution of equation (6) is stable, our numerical code should reproduce it. We conducted two tests. For the first test we desired, as in the example above, that the forced equation had a stationary solution. Of course the solution needs to be nontrivial (non-constant) is space and must satisfy the boundary conditions. For the second test we created a nonstationary solution by multiplying the stationary one by an oscillating function of time. The two cases are represented by a single formula

$$u \equiv v(x, y, t) = A[1 + k \sin(\omega t)]\, x^4 (x - L)^4 y^4 (y - L)^4 \tag{8}$$

where $k = 0$ gives the stationary function and $k \neq 0$ the nonstationary one. Write equation (6) as

$$\partial_t u = \mathrm{RHS} + f(x, y, t)\,, \tag{9}$$

where RHS means the right-hand side of (1). Substituting (8) into (9) we get

$$f(x, y, t) = \partial_t v(x, y, t) - \mathrm{RHS}[v(x, y, t)]\,. \tag{10}$$

Consider the stationary case first ($k = 0$). The shape of the function (8) and that of the force function

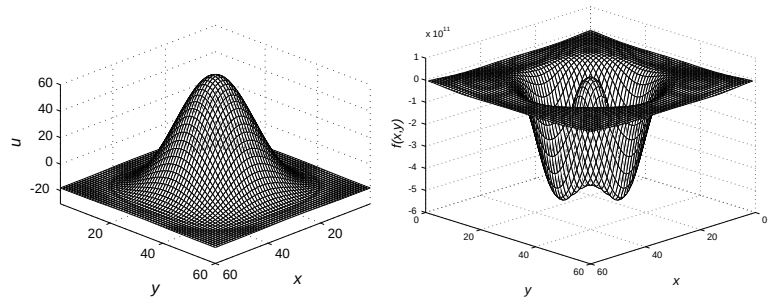

Fig. 1. The exact solution (8) ($k = 0$) [left]. The force function (10) [right]. $A = 1.3 \cdot 10^{13}$, $L = 0.4$. The axes show the number of grid points.

are shown in Fig. 1. The corresponding numerical solution is shown in Fig. 2. It evolves from the initial condition chosen as a dome of the same shape as (8) (with $k = 0$) but with half the height. Note the large time interval between the second and third snapshots. The time step in the computational experiment is $\Delta t = 5 \cdot 10^{-13}$. The settling into the stationary regime is also illustrated by Fig. 3 [left] showing the evolution of the maximum of u taking place in the centre of the domain. Stability of the solution is evident from the experiment. A slight drift of the curve occurs because the stationary

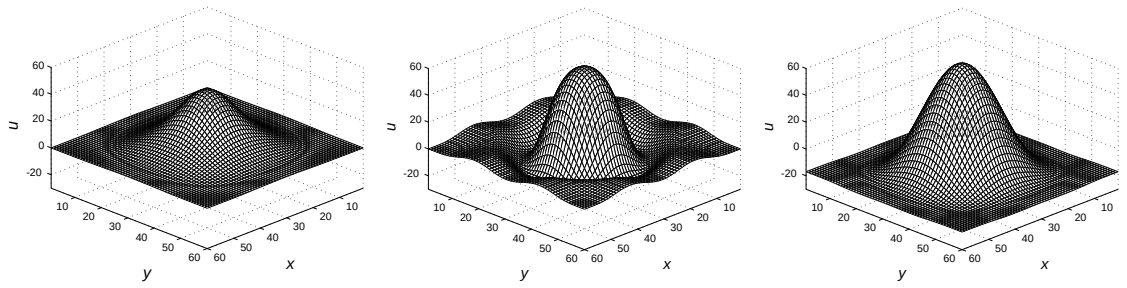

Fig. 2. Evolution of the numerical solution towards the stationary regime. $t = 1.1 \cdot 10^{-11}$, $1.24 \cdot 10^{-10}$, $20.0 \cdot 10^{-10}$. The equation coefficients $g = 10$, $b = c = 1$.

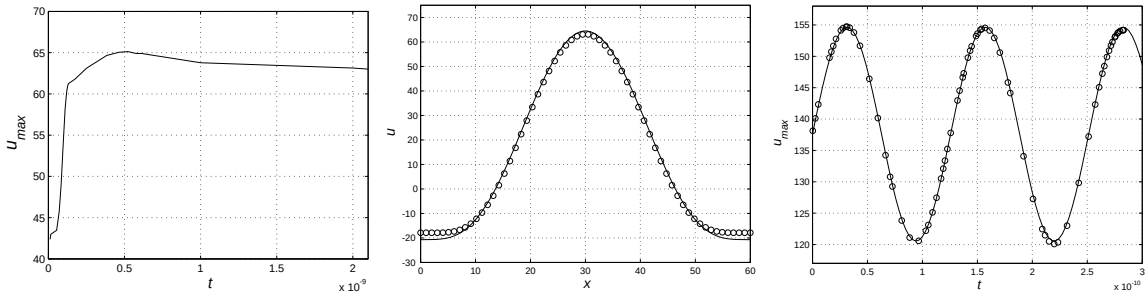

Fig. 3. Evolution of the maximum value of u [left]. The exact and numerical (circles) profiles in the stationary regime [centre]. The exact and numerical (circles) solutions in the oscillatory regime [right].

solution is neutrally stable. In other words, equation (9) allows a family of solutions $u = v(x, y, t) + C$ with different constants C. Inevitable numerical errors appear as a mismatch between the exact force $f(x, y, t)$ and discretized RHS in (9) causing the drift from one stationary solution to another. Fig. 3 [centre] presents the settled profile of the solution along the middle of the domain, $y = 0.2$ (grid line 30), demonstrating a good correspondence except near the edges. Our second test reproduced the oscillating solution (8) with $k = 0.2$, $\omega = 10^{11}$ and the other parameters being the same as before (the time step $\Delta t = 2 \cdot 10^{-14}$). Fig. 3 [right] displays the evolution of the maximum of u. The numerical and exact solutions are close and, again, there is a slow drift due to the neutral stability.

4. Conclusions

We derived the forced phase equation, presented selected exact solutions, and proved their stability by the numerical experiments. The experiments show satisfactory performance of the numerical code. We currently aim at using a finer 90×90 grid, which is, of course, more time-consuming, and also plan to use more sophisticated numerical approaches such as the one-dimensional integrated radial basis function network (1D-IRBFN) method.

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