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A high-order compact integrated-RBF scheme for time-dependent problems

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Abstract

This paper presents a high-order approximation scheme based on compact integrated radial basis function (RBF) stencils and second-order Adams-Bashforth/Crank-Nicolson algorithms for solving time-dependent problems. We employ compact integrated-RBF stencils, where the RBF approximations are locally constructed through integration and expressed in terms of nodal values of the function and its derivatives, to discretise the spatial derivatives in the governing equations. We adopt the Adams-Bashforth and Crank-Nicolson algorithms, which are second-order accurate, to discretise the temporal derivatives. Numerical investigations in several analytic test problems show that the proposed scheme is stable and high-order accurate.

Keywords: time-dependent problems, compact integrated-RBF stencils, high-order approximations

Introduction

High-order approximation schemes have the ability to provide efficient solutions to time-dependent differential problems. A high level of accuracy can be achieved using a relatively coarse discretisation. Many types of high-order schemes have been reported in the literature. For example, in solving the heat equation, Kouatchou (2001) combined a high-order compact finite difference approximation and collocation techniques. Sun and Zhang (2003) proposed a high-order compact boundary value method. Gupta, Manohar, and Stephenson (2005) presented a high-order finite-difference (FD) approximation defined on a square mesh stencil using nine node points. For Burgers' equation, introduced originally by Bateman (1915), Hassanien, Salama, and Hosham (2005) proposed a fourth-order FDM based on two-level three-point of order 2 in time and 4 in space while Zhu and Wang (2009) presented a method based on the cubic B-spline quasi-interpolation. Recently, Hosseini and Hashemi (2011) presented a local-RBF meshless method dealing with several initial and boundary conditions. It is noted that Burgers' equation is considered as a good means of verifying new numerical methods in CFD (Caldwell, Wanless, and Cook, 1987; Iskander and Mohsen, 1992) because the equation contains both the convection and diffusion terms.

Radial basis function networks (RBFNs) have emerged as a powerful approximation tool. The application of RBFNs for the solution of ordinary (ODEs) and partial (PDEs) differential equations was first presented by Kansa (1990). Mai-Duy and Tran-Cong (2001) proposed the use of integration, instead of the usual differentiation, to construct the RBFN expressions (IRBFNs) in order to avoid the reduction of convergence rate. IRBFNs were developed into global one-dimensional forms (1D-IRBF) for second- and fourth-order PDEs (Mai-Duy and Tanner, 2007) and compact local forms for second-order elliptic problems (Mai-Duy and Tran-Cong, 2011). For the latter, the information about the governing equation is also included in local approximations to enhance their accuracy.

In this paper, we present a high-order discretisation scheme based on compact integrated-RBF stencils and second-order Adams-Bashforth/Crank-Nicolson algorithms for solving time-dependent problems, where emphasis is placed on the treatment of the extra information in the compact stencils. The proposed scheme is shown to be stable and high-order accurate. The remainder of the paper is organised as follows. Section 2 briefly outlines several time-dependent equations. The proposed compact integrated-RBF scheme is described in Section 3. In Section 4, numerical results are presented and compared with some published solutions, where appropriate. Section 5 concludes the paper.

Problem formulations

In this paper, two types of time-dependent equation, namely a heat equation and the Burgers' equation, are considered.

Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad a \le x \le b, \ t \ge 0, \tag{1}$$

$$u(x,0) = u_0(x), \ a < x < b, \tag{2}$$

$$u(a,t) = u_{\Gamma_1}(t) \text{ and } u(b,t) = u_{\Gamma_2}(t), \ t \ge 0. \tag{3}$$

where u and t are the field variable and time, respectively; and u_0, u_{Γ_1} and u_{Γ_2} prescribed functions.

Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{\text{Re}}\frac{\partial^2 u}{\partial x^2}, \ a \le x \le b, \ t \ge 0, \tag{4}$$

$$u(x,0) = u_0(x), \ a \le x \le b, \tag{5}$$

$$u(a,t) = u_{\Gamma_1}(t) \text{ and } u(b,t) = u_{\Gamma_2}(t), \ t > 0, \tag{6}$$

where $\text{Re} > 0$ is the Reynolds number; and $u_0(x)$, $u_{\Gamma_1}(t)$ and $u_{\Gamma_2}(t)$ prescribed functions.

Numerical formulations

Temporal discretisation

Heat equation

The temporal discretisation of (1) with a Crank-Nicolson scheme gives

$$\frac{u^n - u^{n-1}}{\Delta t} = \frac{1}{2}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)^n + \left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1}\right], \tag{7}$$

where the superscript n denotes the current time step. It can be rewritten as

$$u^n - \frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^n = u^{n-1} + \frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^{n-1}. \quad (8)$$

Burgers' equation

The temporal discretisation of (4) with an Adams-Bashforth scheme for the convection term and a Crank-Nicolson scheme for the diffusion term gives

$$\frac{u^n - u^{n-1}}{\Delta t} + \left[\frac{3}{2}\left(u\frac{\partial u}{\partial x}\right)^{n-1} - \frac{1}{2}\left(u\frac{\partial u}{\partial x}\right)^{n-2}\right] = \frac{1}{2\,\mathrm{Re}}\left(\frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 u^{n-1}}{\partial x^2}\right) \quad (9)$$

or

$$u^n - \frac{\Delta t}{2\,\mathrm{Re}}\frac{\partial^2 u^n}{\partial x^2} = u^{n-1} + \frac{\Delta t}{2\,\mathrm{Re}}\frac{\partial^2 u^{n-1}}{\partial x^2} - \Delta t\left[\frac{3}{2}\left(u\frac{\partial u}{\partial x}\right)^{n-1} - \frac{1}{2}\left(u\frac{\partial u}{\partial x}\right)^{n-2}\right]. \quad (10)$$

Spatial discretisation

Our previous work (Thai-Quang, Le-Cao, Mai-Duy and Tran-Cong, 2012) provided the details of a compact integrated-RBF approximation to the second-order spatial derivatives over a 3-point stencil (Figure 1). Its formulation can be summarised as follows

$$\frac{du(\eta)}{d\eta} = \sum_{i=1}^{3}\frac{d\varphi_i(\eta)}{d\eta}u_i + \frac{d\varphi_4(\eta)}{d\eta}\frac{d^2u_1}{d\eta^2} + \frac{d\varphi_5(\eta)}{d\eta}\frac{d^2u_3}{d\eta^2} \quad (11)$$

$$\frac{d^2u\ (\eta)}{d\eta^2} = \sum_{i=1}^{3}\frac{d^2\varphi_i(\eta)}{d\eta^2}u_i + \frac{d^2\varphi_4(\eta)}{d\eta^2}\frac{d^2u_1}{d\eta^2} + \frac{d^2\varphi_5(\eta)}{d\eta^2}\frac{d^2u_3}{d\eta^2} \quad (12)$$

where $\eta_1 \leq \eta \leq \eta_3$.

Temporal - spatial discretisation

We consider values of u^n and $\frac{\partial^2 u^n}{\partial x^2}$ at each node as two independent unknowns. As a result, two algebraic equations need be established at each node.

Consider a node x_i. The first equation is obtained by collocating the governing equation, i.e. (8) in the heat equation and (10) in the Burgers equation, at $x = x_i$.

The second equation is derived from (12) for a stencil associated with x_i. Collocating (12) at the central node of a stencil yields

$$\frac{d^2u^n(\eta_2)}{d\eta^2} = \sum_{i=1}^{3}\frac{d^2\varphi_i(\eta_2)}{d\eta^2}u_i^n + \frac{d^2\varphi_4(\eta_2)}{d\eta^2}\frac{d^2u_1^n}{d\eta^2} + \frac{d^2\varphi_5(\eta_2)}{d\eta^2}\frac{d^2u_3^n}{d\eta^2} \quad (13)$$

where $\eta_2 = x_i$. It is noted that, in our previous work (Thai-Quang, Le-Cao, Mai-Duy and Tran-Cong, 2012), values of the second-order derivative at two extreme nodes of a compact stencil are treated like known values using nodal function values at time step $(n-1)$ and nodal second-order derivative values at time step $(n-2)$.

The present final system of the equations is thus solved for the values of the field variable and its second-order derivative at the interior grid nodes.

Numerical examples

It has been accepted that, among RBFs, the multiquadric (MQ) function tends to result in the most accurate approximation (Franke, 1982). We choose MQ as the basis function in the present calculations

$$G_i(x) = \sqrt{(x-c_i)^2 + a_i^2} \tag{14}$$

where c_i and a_i are the centre and width of the i th MQ, respectively. For each stencil, the set of nodal points is taken to be the set of MQ centres. We simply choose the MQ width as $a_i = \beta h_i$ in which β is a given positive number and h_i the distance between the i th node and its nearest neighbour. The value of $\beta = 20$ is used for all calculations in this paper. We assess the performance of the proposed scheme through the following measures:

(i) the root mean square error (RMS) defined as

$$RMS = \frac{\sqrt{\sum_{i=1}^{N}(u_i - \bar{u}_i)^2}}{\sqrt{N}}, \tag{15}$$

where N is the number of nodes over the whole domain and $\bar{u}$ is the exact solution,
(ii) the convergence rate α with respective to grid refinement defined through $Ne \approx O(h^{\alpha})$ as

$$\alpha = \frac{\log\left(Ne^{(r)} / Ne^{(s)}\right)}{\log\left(h^{(r)} / h^{(s)}\right)}, \tag{16}$$

where h is the grid size; and the superscripts r and s are used to indicate the data obtained from the r and s th calculations ($r < s$), respectively.

Heat equation

By selecting this problem, the performance of the proposed scheme can be investigated for the diffusion term only. Consider equation (1) on the segment [0, π] with the initial and boundary conditions $u(x,0) = \sin(2x),\ 0 < x < \pi$ and $u(0,t) = u(\pi,t) = 0,\quad t \geq 0$, respectively. The exact solution of this problem can be verified to be $\bar{u}(x,t) = \sin(2x)e^{-4t}$.

Firstly, the spatial accuracy of the proposed scheme is tested on various uniform grids $N = \{11,21,...101\}$. We employ here a small time step, $\Delta t = 0.001$, to avoid the effect of the approximate error in time. Figure 2 shows the effect of the grid size h on the accuracy of the solution computed at $t = 1$, from which one can see that the solution converges as $O(h^{3.4})$ (i.e. of more than third-order accuracy in space). Figure 3 shows the computed and exact solutions, where good agreement is clearly observed.

Secondly, we test the temporal accuracy of our scheme through a set of time step $\Delta t = \left\{\frac{1}{100}, \frac{1}{90}, ... \frac{1}{10}\right\}$. A fine grid of $h = 0.0157$ (i.e. 201 grid points) is taken so that the approximate error in space is much smaller than the time splitting error. The error of the solution at

$t = 1$ is shown in Figure 4 as a function of the time step Δt. It can be seen that our scheme obtains second-order accuracy in time. This result is fully expected as second-order approximation algorithms for the discretisation of time derivative terms are adopted in the present scheme.

Burgers' equation

This equation involves the convection and diffusion terms. Consider a particular solution, namely shock wave propagation, of Burgers' equation (4) on the segment 0 ≤ x ≤ 1, t ≥ 0 in the form (Hassanien, Salama, and Hosham, 2005)

$$\bar{u}(x,t) = \frac{\left[\alpha_0 + \mu_0 + (\mu_0 - \alpha_0)\exp(\eta)\right]}{1+\exp(\eta)}, \tag{17}$$

where $\eta = \alpha_0 \,\mathrm{Re}(x - \mu_0 t - \beta_0)$, $\alpha_0 = 0.4$, $\beta_0 = 0.125$, $\mu_0 = 0.6$, $\mathrm{Re} = 100$.

The initial and boundary conditions can be derived from the analytic solution (17). The calculation is carried out at $\mathrm{Re} = 100$ on the grid of $N = 37$. Results obtained are presented in Table 1.The exact solution and some other solutions presented in (Ali, Gardner, and Gardner, 2005; Dogan, 2004; Dag, Irk, and Saka, 2005; Hassanien, Salama, and Hosham, 2005) are also included. It can be seen that the present solution is in good agreement with the other solutions. Figure 5 illustrates the computed solution at different times, which are indistinguishable from the exact solutions.

In order to study the convergence of the solution with grid refinement, the calculations are carried out on a set of uniform grids $N = \{11, 21, \ldots 101\}$. The time step Δt is required to be small enough at which the error caused by the temporal discretisation can be negligible. In the present study, the time step $\Delta t = 10^{-5}$ is chosen. The errors of the solution against different grid sizes at the time $t = 0.5$ are displayed in Figure 6, where the solution converges very fast at a high rate of 4.47.

Conclusions

In this paper, a new discretisation scheme for time-dependent problems is presented. The present approximations are based on 3-node stencils, resulting in a sparse system matrix, while Adams-Bashforth/Crank Nicolson algorithms and IRBF-based compact approximations are employed to yield a high-order accurate solution in time and space, respectively. Several test problems are considered to verify the present scheme.

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Table 1. Shock wave propagation, grid N=37, Re=100, t=0.5: Comparison between exact and computed solutions

x	Exact solution	Present $\Delta t = 0.05$	Ali et al. $\Delta t = 0.025$	Dogan $\Delta t = 0.05$	Dag et al. $\Delta t = 0.025$	Hassanien et al. $\Delta t = 0.01$
0.000	1.0	1.0	1.0	1.0	1.0	1.0
0.056	1.0	1.0	1.0	1.0	1.0	1.0
0.111	1.0	1.0	1.0	1.0	1.0	1.0
0.167	1.0	1.0	1.0	1.0	1.0	1.0
0.222	1.0	1.0	1.0	1.0	1.0	1.0
0.278	0.998	0.998	0.999	0.999	0.999	0.998
0.333	0.980	0.980	0.985	0.994	0.986	0.982
0.389	0.847	0.850	0.847	0.848	0.850	0.849
0.444	0.452	0.449	0.452	0.407	0.448	0.455
0.500	0.238	0.238	0.238	0.232	0.236	0.239
0.556	0.204	0.204	0.204	0.204	0.204	0.204
0.611	0.2	0.2	0.2	0.2	0.2	0.2
0.667	0.2	0.2	0.2	0.2	0.2	0.2
0.722	0.2	0.2	0.2	0.2	0.2	0.2
0.778	0.2	0.2	0.2	0.2	0.2	0.2
0.833	0.2	0.2	0.2	0.2	0.2	0.2
0.889	0.2	0.2	0.2	0.2	0.2	0.2
0.944	0.2	0.2	0.2	0.2	0.2	0.2
1.000	0.2	0.2	0.2	0.2	0.2	0.2

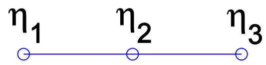

Figure 1.Compact local 3-point 1D-IRBF stencil

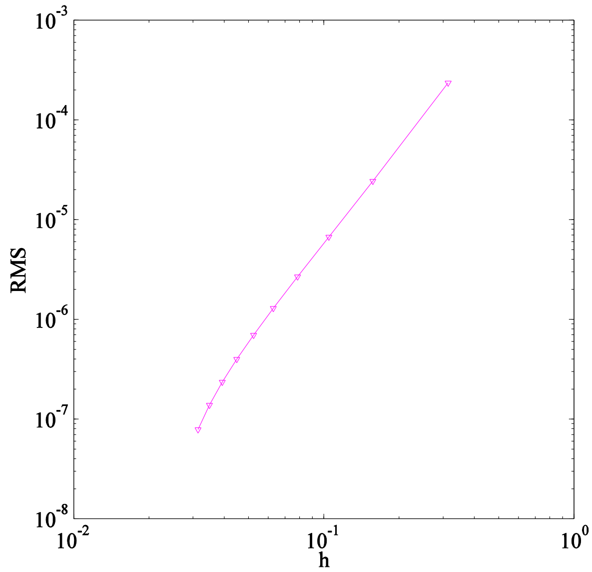

Figure 2. Heat equation, N={11, 21, ...101}, Δt=0.001, t=1: The effect of grid size h on the solution accuracy by the proposed scheme. The solution converges as $O(h^{3.4})$

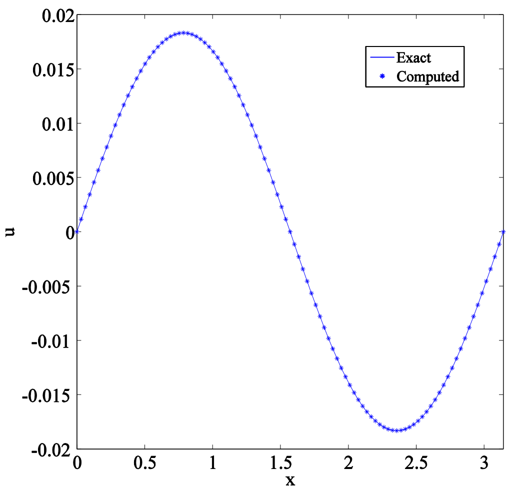

Figure 3. Heat equation, N=101, Δt=0.001, t=1: Variations of the computed and exact solutions

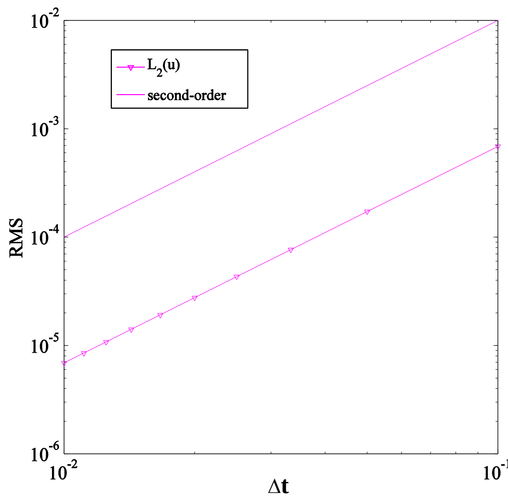

Figure 4. Heat equation, N=201, Δt={1/100, 1/90, ...1/10}, t=1: The effect of time step Δt on the solution accuracy by the proposed scheme. The solution converges as $O(\Delta t^2)$

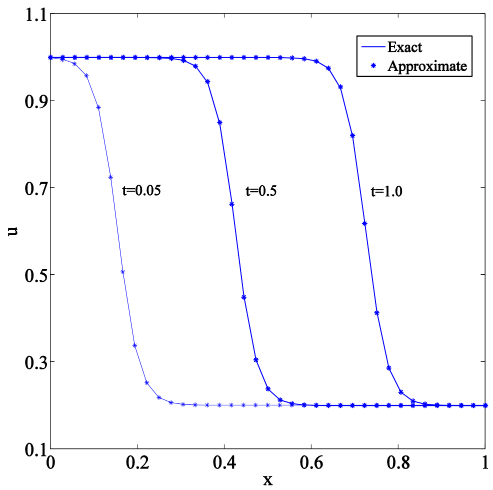

Figure 5. Shock wave propagation, N=37, Re=100: Profiles of the computed and exact solutions at different times

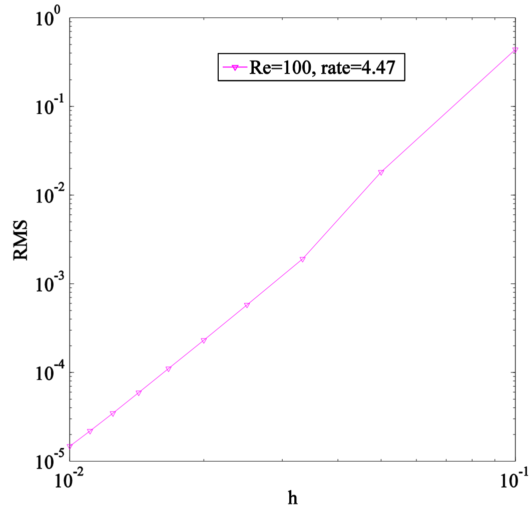

Figure 6. Shock wave propagation, N={11,21, ...101}, Re=100, $\Delta t=10^{-5}$, t=0.5: The effect of grid size h on the solution accuracy by the proposed scheme. The solution converges as $O(h^{4.47})$