A new high-order time-kernel BIEM for the Burgers equation

N. Mai-Duy[1a], T. Tran-Cong[1b], R.I. Tanner[2]

[1a] maiduy@usq.edu.au and [1b] trancong@usq.edu.au

Faculty of Engineering and Surveying

The University of Southern Queensland, QLD 4350, Australia

[2] rit@aeromech.usyd.edu.au

School of AMME

The University of Sydney, NSW 2006, Australia

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In this paper, a high-order interpolation scheme, namely integrated radial-basis-function networks (IRBFNs), is introduced into the time-kernel boundary-integral-equation method (BIEM) to approximate the unknown functions in boundary and volume integrals for the solution of the Burgers equation. All relevant integrals are written in terms of nodal variable values. Solutions over the temporal domain can be obtained at once rather than step by step as in the case of conventional BIEMs.

Parabolic differential equations have been employed in a variety of engineering problems. Solutions to these equations can be found by means of numerical discretization methods such as BIEMs, finite-difference (FDMs), finite-element (FEMs) and finite-volume (FVMs) methods. For BIEMs (e.g. [1,2]), there are several approaches proposed to deal with a time-derivative term. Based on the criterion of fundamental solutions used, they can be classified into two groups. The first group employs time-dependent fundamental solutions, i.e. time derivatives enter the integral representation through the kernel functions, which allows the process of discretization in time and space to be conducted in a similar fashion. The second group employs stationary fundamental solutions. Some additional treatments for time derivatives are thus required; they generally fall into one of two categories: Laplace transforms and finite-difference schemes.

In the context of time-kernel BIEMs, there are relatively few papers on using high-order interpolation schemes to approximate the unknown functions with respect to time. The case of using quadratic functional variation was reported in [2]. Recently, Grigoriev and Dargush [3] employed quartic interpolation functions, and their obtained results indicated a significant improvement in accuracy, convergence rate and error distribution.

Radial-basis-function networks have found a wide range of applications in the field of numerical analysis. These networks exhibit good approximation properties. For example, it has been proved that RBFNs are capable of representing any continuous function to a desired level of accuracy by increasing the number of hidden neurons (universal approximation) [4]. Madych and Nelson [5] showed that the multiquadric (MQ) interpolation scheme converges exponentially with respect to the number of data points used. It was found that IRBFNs have higher approximation power than differentiated RBFNs [6].

In the present work, instead of using high-order Lagrange polynomials such as quadratic and quartic interpolation functions, the proposed method employs MQ-IRBFNs to represent the unknown functions in boundary and volume integrals. Numerical implementations of ordinary and double integrals involving time in the presence of IRBFNs are presented in detail. All relevant integrals are written in terms of nodal variable values. The proposed method is verified through the solution of diffusion and convection-diffusion problems. Numerical results obtained show that the IRBFN-BIEM attains a significant improvement in accuracy over low-order time-kernel BIEMs and FDMs (Tables 1 & 2).

	u (error %)				
t	FDM-CN	FDM-PI	FDM-WT	Present	Exact
0.025	0.5637(12.35)	0.6888(7.09)	0.6807(5.84)	0.6432(0.00)	0.6432
0.050	0.5440(9.69)	0.5330(7.47)	0.5286(6.58)	0.4959(0.01)	0.4959
0.075	0.3493(9.68)	0.4226(9.27)	0.4188(8.29)	0.3867(0.01)	0.3868
0.100	0.3313(9.66)	0.3376(11.76)	0.3341(10.58)	0.3021(0.01)	0.3021
0.125	0.2117(10.31)	0.2705(14.58)	0.2671(13.14)	0.2360(0.01)	0.2360
0.150	0.2038(10.50)	0.2169(17.60)	0.2137(15.86)	0.1844(0.01)	0.1844
0.175	0.1270(11.84)	0.1740(20.74)	0.1710(18.67)	0.1441(0.01)	0.1441
0.200	0.1262(12.10)	0.1396(20.99)	0.1369(21.57)	0.1126(0.01)	0.1126
0.225	0.0756(14.12)	0.1120(27.33)	0.1096(24.54)	0.0880(0.01)	0.0880
0.250	0.0787(14.47)	0.0899(30.76)	0.0877(27.58)	0.0687(0.01)	0.0687

Table 1. Diffusion problem, $0 \le x \le 1, 0 \le t \le 0.25, \Delta x = 0.1, \Delta t = 0.025$: Temperature at the centre of the slab obtained by the present method and various FDMs. The latter which use the same time step and finer spatial discretization ($\Delta x = 0.05$) are extracted from the paper of Haberland and Lahrmann [7] in which CN stands for Crank-Nicolson, PI: pure implicit and WT: weighted time step. The maximum error is 0.01% for IRBFN-BIEM while they are 14.47%, 30.76% and 27.58% for FDM-CN, FDM-PI and FDM-WI, respectively.

	u (error %)					
	$t = 1.2$			$t = 3$		
x	GBIEM	Present	Exact	GBIEM	Present	Exact
0.20	0.13173(0.62)	0.13092(0.00)	0.13092	0.06036(0.44)	0.06010(0.01)	0.06009
0.40	0.26285(0.60)	0.26127(0.01)	0.26128	0.12068(0.43)	0.12015(0.00)	0.12016
0.60	0.39264(0.56)	0.39043(0.00)	0.39044	0.18095(0.43)	0.18018(0.00)	0.18018
0.80	0.52012(0.50)	0.51756(0.01)	0.51753	0.23906(0.18)	0.23861(0.00)	0.23863
0.90	0.57950(0.29)	0.57780(0.00)	0.57781	0.23766(1.63)	0.24158(0.00)	0.24159
0.92	0.58466(0.01)	0.58482(0.02)	0.58472	0.22050(2.48)	0.22616(0.00)	0.22612
0.94	0.57336(0.77)	0.57760(0.03)	0.57779	0.18997(3.51)	0.19696(0.01)	0.19690
0.96	0.51253(2.42)	0.52551(0.05)	0.52524	0.14228(4.58)	0.14915(0.01)	0.14911
0.98	0.33294(5.04)	0.35023(0.11)	0.35060	0.07698(5.41)	0.08140(0.01)	0.08139

Table 2. Convection-diffusion problem, $\text{Re} = 100, 0 \le t \le 3, 0 \le x \le 1, \Delta x = 0.02, \Delta t = 0.1$: Solution profiles at some time levels. Results by the generalized BIEM (GBIEM) using linear and constant elements, $\Delta x = 0.01$ and $\Delta t = 0.01$ [8] are also included. The proposed method yields a very high level of accuracy and its errors do not accumulate in time.

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